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610. A MODIFICATION OF THE BIRKHOFF-YOUNG QUADRATURE FORMULA FOR ANALYTICAL FUNCTIONS

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Dedicated to Professor D. S. Mitrinović on his 70-th birthday

1. Introduction. In methods of numerical integration we use a linear combination of function values at points usually chosen so as to lie in the interval of integration. If $x \mapsto f(x)$ is a real-valued function, then in numerical quadrature rules function values for complex arguments can be used. Let function $z \mapsto f(z)$ be a regular analytic function in a domain D which contains the interval of integration. BIRKHOFF and YOUNG [1] derived the five-point formula (named BY formula)

$$\int_{-1}^{1} f(z)\,\mathrm{d}z = \frac{8}{5} f(0) + \frac{4}{15}\big(f(1) + f(-1)\big) - \frac{1}{15}\big(f(i) + f(-i)\big) + R_{BY}, \tag{1}$$

or more generally

$$\int_{z_0-h}^{z_0+h} f(z)\,\mathrm{d}z = \frac{8}{5} h f(z_0) + \frac{4}{15} h\big(f(z_0+h) + f(z_0+h)\big) - \frac{1}{15} h\big(f(z_0+ih) + f(z_0-ih)\big) + R_{BY}, \tag{2}$$

where the error term R_{BY} vanishes on polynomials of degree 5 or less. As it can be seen, in these quadrature formulas we take function values in the vertices and in the center of the square, where two vertices coincide with the ends of the integration interval.

The estimation of the error term R_{BY}, which is related to equation (2), is given in [2] in the form

$$|R_{BY}| \leq \frac{1}{1890} |h|^7 \max_{z \in S} |f^{(6)}(z)|, \tag{3}$$

where S is a square with vertices defined as arguments in (2). LETHER [3] pointed out that the three-point GAUSS-LEGENDRE quadrature formula (GL) is more precise than (1) and he recommended the application of GL for an integration over a straight line segment. LYNESS and DELVES [4], and later LYNESS [5] developed quadrature rules for numerical integration of complex functions. These rules are founded on the integration of the TAYLOR series, where derivations of analytical functions were approximatively obtained by application of the trapezoidal rule on CAUCHY contour integrals [6].

In the present paper the new five-point quadrature rule is derived. This rule appears to give more accurate results than BY and GL formulas.

2. The modified Birkhoff-Young formula. Let us consider an integral $\int_{-1}^{1} f(z)\,\mathrm{d}z$, where $z \mapsto f(z)$ is an analytical function in the square whose vertices are 1, i, -1, $-i$. If we integrate the TAYLOR expansion of f around $z=0$, we obtain

$$\int_{-1}^{1} f(z)\,\mathrm{d}z = 2\left(f(0)+\frac{f''(0)}{3!}+\frac{f^{(4)}(0)}{5!}+\frac{f^{(6)}(0)}{7!}+\cdots\right). \tag{4}$$

Derivatives $f''(0)$ and $f^{(4)}(0)$ can be written in the form [7]

$$\begin{aligned} f''(0) &= \frac{1}{2k^2}\left(f(k)+f(-k)-f(ik)-f(-ik)\right) \\ &\quad -2\left(\frac{k^4}{6!}f^{(6)}(0)+\frac{k^8}{10!}f^{(10)}(0)+\frac{k^{12}}{14!}f^{(14)}(0)+\cdots\right), \end{aligned} \tag{5}$$

$$\begin{aligned} f^{(4)}(0) &= \frac{6}{k^4}\left(f(k)+f(-k)+f(ik)+f(-ik)-4f(0)\right) \\ &\quad -24\left(\frac{k^4}{8!}f^{(8)}(0)+\frac{k^8}{12!}f^{(12)}(0)+\cdots\right), \end{aligned} \tag{6}$$

where k, $-k$, ik and $-ik\,(k \leqq 1)$ are vertices of the square in which we calculate a function f for an approximate computation of derivatives.

Introducing (5) and (6) in (4), we obtain an integration rule

$$\begin{aligned} \int_{-1}^{1} f(z)\,\mathrm{d}z &= 2\left(1-\frac{1}{5k^4}\right)f(0)+\left(\frac{1}{6k^2}+\frac{1}{10k^4}\right)\left(f(k)+f(-k)\right) \\ &\quad +\left(-\frac{1}{6k^2}+\frac{1}{10k^4}\right)\left(f(ki)+f(-ki)\right)+R, \end{aligned} \tag{7}$$

with the error term

$$R=\left(-\frac{2}{3}\cdot\frac{1}{6!}k^4+\frac{2}{7!}\right)f^{(6)}(0)+\left(\frac{2}{9!}-\frac{2}{5\cdot 8!}k^4\right)f^{(8)}(0)+\cdots$$

If we put $k=1$ in (7), we derive the BY formula (1) with a slightly changed error term

$$\begin{aligned} \int_{-1}^{1} f(z)\,\mathrm{d}z &= \frac{8}{5}f(0)+\frac{4}{15}\left(f(1)+f(-1)\right)-\frac{1}{15}\left(f(i)+f(-i)\right) \\ &\quad -\frac{1}{1890}f^{(6)}(0)-\frac{1}{226800}f^{(8)}(0)+\cdots \end{aligned}$$

It is interesting that for $-\frac{1}{6k^2}+\frac{1}{10k^4}=0$, i.e. for $k=\sqrt{0.6}$, when the coefficient of $f(ki)+f(-ki)$ vaniches, we obtain three-point GL formula

$$\int_{-1}^{1} f(z)\,\mathrm{d}z=\frac{8}{9}f(0)+\frac{5}{9}\left(f(\sqrt{0.6})+f(-\sqrt{0.6})\right)$$
$$+\frac{1}{15750}f^{(6)}(0)-\frac{1}{226800}f^{(8)}(0)+\cdots$$

3. A modified formula of the maximum accuracy. In the aim of obtaining a five-point quadrature rule which will be exact for as high degree polynomials as possible, we can choose the coefficient of $f^{(6)}(0)$ in (7) to be zero. Thus, we obtain $k=\sqrt[4]{3/7}$ and from (7) we obtain the following rule (named (MF)):

$$\int_{-1}^{1} f(z)\,\mathrm{d}z=\frac{16}{15}f(0)+\frac{1}{6}\left(\frac{7}{5}+\sqrt{\frac{7}{3}}\right)\left(f\left(\sqrt[4]{\frac{3}{7}}\right)+f\left(-\sqrt[4]{\frac{3}{7}}\right)\right) \tag{8}$$
$$+\frac{1}{6}\left(\frac{7}{5}-\sqrt{\frac{7}{3}}\right)\left(f\left(i\sqrt[4]{\frac{3}{7}}\right)+f\left(-i\sqrt[4]{\frac{3}{7}}\right)\right)+R_{MF},$$

with error term

$$R_{MF}=\frac{1}{793800}f^{(8)}(0)+\frac{1}{61122600}f^{(10)}(0)+\cdots$$

As it can be seen, the error is considerably smaller than errors in BY and three-point GL quadrature formulas. If the interval of integration $[-1,1]$ is transformed in the complex line segment with end points z_0-h and z_0+h, equation (8) can be shown in the form

$$\int_{z_0-h}^{z_0+h} f(z)\,\mathrm{d}z=\frac{16}{15}hf(z_0)+\frac{h}{6}\left(\frac{7}{5}+\sqrt{\frac{7}{3}}\right)\left(f\left(z_0+h\sqrt[4]{\frac{3}{7}}\right)+f\left(z_0-h\sqrt[4]{\frac{3}{7}}\right)\right)$$
$$+\frac{h}{6}\left(\frac{7}{5}-\sqrt{\frac{7}{3}}\right)\left(f\left(z_0+ih\sqrt[4]{\frac{3}{7}}\right)+f\left(z_0-ih\sqrt[4]{\frac{3}{7}}\right)\right)$$
$$+\frac{h^9}{793800}f^{(8)}(z_0)+\frac{h^{11}}{61122600}f^{(10)}(z_0)+\cdots$$

In table 1 first terms in error of BY, GL and MF formulas are presented. From this table we clearly see an advantage of MF.

Table 1

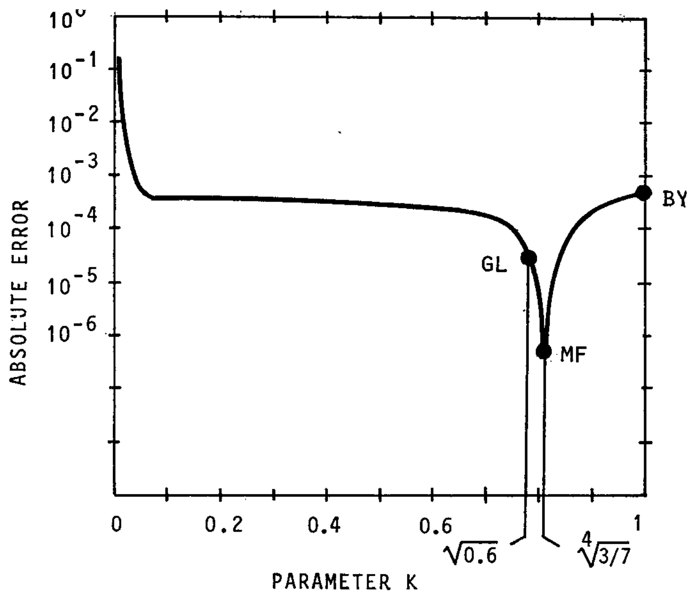
Formula	First term in the error
BY	$-\frac{1}{1890}h^7 f^{(6)}(z_0)$
GL	$\frac{1}{15750}h^7 f^{(6)}(z_0)$
MF	$\frac{1}{793800}h^9 f^{(8)}(z_0)$

Numerical example. $\int_{-1}^{1} e^x \mathrm{d}x$ has been evaluated by use of *BY*, *GL* and *MF* formulas. The exast value of the integral on ten significant digits is 2.350402387. Results are given in table 2. Comparing these results with the estimated errors given in table 1 we observe a good agreement.

Table 2

Formula	Approximate integral	Error
BY	2.350936031	$\sim 5.3 \cdot 10^{-4}$
GL	2.350336929	$\sim 6.5 \cdot 10^{-5}$
MF	2.350401111	$\sim 1.3 \cdot 10^{-6}$

In order to stress the advantage of the *MF*, integral $\int_{-1}^{1} e^x \mathrm{d}x$ was calculated by means of (7) for different values of k, belonging [0,1]. The results are presented on a diagram giving the absolute value of the error in dependence of k. As we may see, in the vicinity of $k=\sqrt[4]{3/7}$ the pronounced minimum connected with the error decrease of the *MF* is obtained.



The *MF* could by simply used to integrate the analytical functions with respect to the arbitrary straight line segment in the complex plane, with end points z_0 and z_n. If this segment is subdivided into n parts, the integral related to $[z_{j-1}, z_j]$ could be expressed in the form

$$\int_{z_{j-1}}^{z_j} f(z)\,\mathrm{d}z = a_j \int_{-1}^{1} f(a_j t + b_j)\,\mathrm{d}t$$

$$\approx I_j = a_j \left(\frac{16}{15} f(b_j) + \frac{1}{6}\left(\frac{7}{5} + \sqrt{\frac{7}{3}}\right)\left(f(b_j + \alpha a_j) + f(b_j - \alpha a_j)\right)\right.$$

$$\left. + \frac{1}{6}\left(\frac{7}{5} - \sqrt{\frac{7}{3}}\right)\left(f(b_j + \alpha a_j i) + f(b_j - \alpha a_j i)\right)\right),$$

where $a_j = \frac{z_j - z_{j-1}}{2}$, $b_j = \frac{z_j + z_{j-1}}{2}$, $\alpha = \sqrt[4]{\frac{3}{7}}$.

On the basis of that we have

$$\int_{z_0}^{z_n} f(z)\,\mathrm{d}z \approx \sum_{j=1}^{n} I_j.$$

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