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608. RELATIVE SINGULAR VALUES OF LINEAR TRANSFORMATIONS

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To Professor D. S. Mitrinović, with my admiration

This exposition intends to give some applications of properties of proper values and singular values of linear transformations on a unitary space to relative singular values of linear transformations. We shall give a few samples.

1. Definitions and Notations. A unitary spece of dimension n will be denoted by E_n. Vectors will be indicated by Greek letters and complex numbers by latin letters. Most standard notations will be used. Let A be a linear transformation on E_n. Then the positive square roots of the proper values of A^*A are called absolute singular values of A or simply singular values of A. Let H be a Hermitian transformation on E_n. Then proper values of A^*HA are called relative singular values of A relative to H. In particular we will study the case that H is a positive (positive definite) transformation. Thus we will use the following definition:

Let H be a positive linear transformation and A any linear transformation on E_n. Then A^*HA is non-negative **[2]**. The positive square roots of proper values of A^*HA are defined to be *relative singular values* of A relative to H.

If $j_p \leqq i_p$ for $p = 1, \ldots, k$, we write $(j_1, \ldots, j_k) \leqq (i_1, \ldots, i_k)$. Given any sequence $i_1 \leqq \cdots \leqq i_k$ of integeres such that $i_p \geqq p$, $p = 1, \ldots, k$, let $(i_1', \ldots, i_k')$ denote the strictly increasing sequence of positive integers such that

(a) $$(i_1', \ldots, i_k') \leqq (i_1, \ldots, i_k),$$

(b) $$(j_1, \ldots, j_k) \leqq (i_1, \ldots, i_k),$$

where $(j_1, \ldots, j_k)$ is a strictly increasing sequence of positive integers which is less than or equal to $(i_1, \ldots, i_k)$. One observes that $(i_1', \ldots, i_k')$ is given by

$$i_k' = i_k, \; i_p' = \min(i_p, \; i_{p+1} - 1) \; (p = k-1, \ldots, 1).$$

2. A Set of Theorems. We shall state the theorems which will be used in this note **[1]**.

(i) Let $i_1 \leqq \cdots \leqq i_k$ and $j_1 \leqq \cdots \leqq j_k$ be sequences of positive integers such that $i_k \leqq n$, $j_k \leqq n$, $i_p + j_j \geqq n + p$, $p = 1, \ldots, k$, and $k \leqq n$. Let $C = A + B$, where A and B are Hermitian transformations on E_n. Suppose

$$a_1 \geqq \cdots \geqq a_n, \; b_1 \geqq \cdots \geqq b_n, \; c_1 \geqq \cdots \geqq c_n$$

are proper values of A, B, and C respectively. Then

$$a'_{i1}+\cdots+a'_{ik}+b'_{j1}+\cdots+b'_{jk}\leqq c_{(i_1+j_1-n)'}+\cdots+c_{(i_k+j_k-n)'},$$

where, for example, $(i_1', \ldots, i_k')$ is the sequence described in § 1.

(ii) Let A and B be non-negative transformations on E_n and $C=AB$. Then the proper values of C are real and non-negative. Let $c_1\geqq\cdots\geqq c_n$, $a_1\geqq\cdots\geqq a_n$, and $b_1\geqq\cdots\geqq b_n$ be respectively proper values of C, A and B. Let $(i_1, \ldots, i_k)$ be a strictly increasing sequence of positive integers such that $i_k\leqq n$. Then.

(1) $$c_{i1}\cdots c_{ik}\leqq a_1\cdots a_k\cdot b_{i1}\cdots b_{ik},$$

(2) $$c_{i1}\cdots c_{ik}\leqq b_1\cdots b_k\cdot a_{i1}\cdots a_{ik}.$$

There are other theorems which may be applied. We shall leave them to the reader.

3. Theorem. *Let A be a linear transformation and H a positive transformation on E_n. Let $a_1\geqq\cdots\geqq a_n$ be singular values of A, $r_1\geqq\cdots\geqq r_n$ be relative singular values of A relative to H, and $h_1\geqq\cdots\geqq h_n$ be proper values of H. Then*

(1) $$r_{i1}^2\cdots r_{ik}^2\leqq a_1^2\cdots a_k^2\cdot h_{i1}\cdots h_{ik},$$

and

(2) $$r_{i1}^2\cdots r_{ik}^2\leqq h_1\cdots h_k\cdot a_{i1}^2\cdots a_{ik}^2,$$

where $(i_1, \ldots, i_k)$ is the sequence described in § 2 (i).

Proof. Even though A^*HA itself does not fit the hypothesis of 2 (i); one observes that AA^*H has the same proper values as A^*HA. Thus the proof is clear.

4. Theorem (Relative Singular Values of a Sum). *Let A and B be linear transformations on E_n. Let $r_1\geqq\cdots\geqq r_n$ and $s_1\geqq\cdots\geqq s_n$ be respectively relative singular values of A and B relative to H, a positive linear transformation on E_n. Let $t_1\geqq\cdots\geqq t_n$ be relative singular values of $A+B$. Then*

$$r_{i_1'}^2+\cdots+r_{i_k'}^2+s_{j_1'}^2+\cdots+s_{j_k'}^2\leqq t_{(i_1+j_1-n)'}+\cdots+t_{(i_k+j_k-n)'},$$

where $(i_1', \ldots, i_k')$ is the sequence described in § 1.

This theorem follows directly from § 2 (i).

5. Theorem (Relative Singular Values of a Product). *Let A and B be linear transformations on E_n. Let $r_1\geqq\cdots\geqq r_n$ be relative singular value of AB relative to H, a positive transformation on E_n. Let $s_1\geqq\cdots\geqq s_n$ be relative singular values of A relative to H and $b\geqq\cdots\geqq b_n$ be absolute singular vplues of B. Then*

(1) $$r_{i1}\cdots r_{ik}\leqq s_1\cdots s_k\, b_{i1}\cdots b_{ik},$$

and

(2) $$r_{i1}\cdots r_{ik}\leqq s_{i1}\cdots s_{ik}\, b_1\cdots b_k,$$

where $(i_1, \ldots, i_k)$ is the sequence described § 2 (i).

Proof. One observes that B^*A^*HAB has the same proper values as $(A^*HA)(BB^*)$. Thus applying 2 (i) the proof is clear.

6. Questions. In this note only a sample set of applications was given. One may obtain other inequalities. If H is any Hermitian transformation, the problem becomes complicated. It would be interesting to study this case. One also can extend the problem to the case that H is non-negative.

Note that in what was studied we didn't obtain inequalities relating relative singular values of A, B, and AB of section 5. Thus, this is left as a question.

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