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607. NOTES ON CONVEX FUNCTIONS II: ON CONTINUOUS LINEAR OPERATORS DEFINED ON A CONE OF CONVEX FUNCTIONS*

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0. In this paper a theorem (Theorem 2) is proved providing necessary and sufficient conditions for the validity of (1) for an arbitrary convex function, A being a linear and continuous operator. Theorems 3 — 5 are equivalent forms of theorem 2 which are more suitable for applications. The applications of the results obtained in this paper will be presented in a few forthcoming notes which will be published under the same title.

1. A large number of theorems from the theory of convex functions are of the from

$$Af \geqq 0 \tag{1}$$

where $x \mapsto f(x)$ is a convex function and A a given linear operator. We must emhasize, at the beginning, that we refer to the continuous function satisfying the JENSEN inequality on a closed and finite segment of the real straight line (see [**1**] p. 17). On one hand a lot of properties of the class of convex functions have form (1) and, on the other hand, the consequences of these properties are also of the form (1). Roughly speaking, operators A appearing in (1) are most often linear discrete operators, linear differential operators, linear integ·al operators as well as their combinations (i.e. operators stemming from the superposition of the previously derived operators). For the sake of illustration we shall mention few examples.

Let nonnegative numbers $p_i (i=1, \ldots, n)$ be such that $\sum_{i=1}^{n} p_i = 1$ and let f be a convex function on segment $[a, b]$. It is well known that the following inequality

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leqq \sum_{i=1}^{n} p_i f(x_i) \tag{2}$$

holds, for the arbitrary points $x_i \in [a, b]$ $(i=1, \ldots, n)$. If the points x_i and the weights p_i are fixed then defining the operator A by

$$Af = \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \tag{3}$$

inequality (2) takes form (1). This is an example of a linear discrete operator. In this case we assume that n is a fixed natural number.

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It is well known that twice differentiable function f is convex if and only if $f'' \geqq 0$. Assuming that $A = \frac{d^2}{dx^2}$ the condition $f'' \geqq 0$ gets form (1). This is an example of a differential linear operator.

The majorization theorem (see **[1]** p. 126) supplies us with the necessary and sufficient condition for validity of the inequality

$$\int_0^1 f(x(t))\, dt \leqq \int_0^1 f(y(t))\, dt \tag{4}$$

for every convex function f. At present we shall consider the linear integral operator A of the form

$$Af = \int_0^1 f(y(t))\, dt - \int_0^1 f(x(t))\, dt. \tag{5}$$

Using (5), condition (4) gets form (1).

Starting, thus, from these examples of linear operators (as well as from several others not quoted here) we arrived at the conclusion that it would be of interest to give the necessary and sufficient conditions for arbitrary linear operator so that (1) holds for any convex function f.

In the present paper we shall give a thoorem relevent to the necessary and sufficient conditions for (1) to be valid for the arbitrary convex function f, as well as several equivalents of it, i.e. consequences. The proof of this theorem of ours is based on papers by K. Toda **[2]** and T. Popoviciu **[3]**. As far as we know in the literature relevant to convex functions such a theorem was not explicitly stated, up to now, though some of its very particular cases were mentioned in **[2]**, **[3]** and **[4]**.

The applications of our theorem, i.e. some of the particular cases of operator A will be presented in a few "notes" from this series.

2. As customary, we shall denote by $C[a, b]$ a set of all functions continuous on the segment $[a, b]$ $(-\infty < a < b < +\infty)$ (f is continuous from the right at a and f is continuous from the left at b). The norm of a function $t \mapsto x(t) \in C[a, b]$ will be defined by

$$\| x \| = \max_{a \leqq t \leqq b} | x(t) |.$$

The sequence $x_n = x_n(t) \in C[a, b]$ will be assumed to converge towards function $x = x(t) \in C[a, b]$ if $\lim_{n \to +\infty} \| x_n - x \| = 0$.

We shall further observe operators whose domain is the space $C[a, b]$. Let A be such an operator. The operator A will be considered as linear if

$$A(px + qy) = pAx + qAy$$

holds for each pair of real numbers p and q and for each pair of functions $x = x(t)$, $y = y(t) \in C[a, b]$.

Let $D \subset R$ and let $S(D)$ be one of the normed subspaces of the space of all real functions defined on D, with the norm $\|x\|_1$ for $x \in S(D)$. For the operator A: $C[a, b] \to S(D)$ we will say that it is continuous at the point $x_0 \in C[a, b]$ if from the condition $\|x_n - x_0\| \to 0$ $(n \to +\infty)$ it follows that $\|Ax_n - Ax_0\|_1 \to 0$ $(n \to +\infty)$ for every sequences of points $x_n \in C[a, b]$.

For the operator A: $C[a, b] \to S(D)$ we shall write $Ax \geqq 0$ if $Ax = F(t) \geqq 0$ for every $t \in D$, where $x = x(t)$ is a given function from $C[a, b]$.

The real function f will be said to be convex on $[a, b]$ if the inequality

$$f(pt_1 + (1-p)t_2) \leqq pf(t_1) + (1-p)f(t_2)$$

is valid for every pair of points $t_1, t_2 \in [a, b]$ and for any $p \in [0, 1]$. The set of all convex functions on $[a, b]$ will be denoted by $K[a, b]$. It is obvious from the foregoing that the inclusion $K[a, b] \subset C[a, b]$ is valid (we consider only those convex functions which are continuous from the right at the point a and which are continuous from the left at the point b).

In the further work the following theorem will be used, which was proved by K. Toda [2] and T. Popoviciu [3] and which reads:

Theorem 1. (a) *Every function of the sequence*

$$G_m(t) = pt + q + \sum_{k=0}^{m} p_k |t - t_k| \qquad (m = 1, 2, \ldots), \tag{6}$$

where $t \in [a, b]$; $p, q \in \mathbf{R}$; $p_k \geqq 0$, $t_k \in [a, b]$ $(k = 0, 1, \ldots, m)$ *is convex on* $[a, b]$.

(b) *Every function f convex on $[a, b]$ is the uniform limit of the sequence G_m of the form* (6) *where* $p, q \in \mathbf{R}$, $p_k \geqq 0$, $t_k \in [a, b]$ $(k = 0, 1, \ldots, m)$.

In the paper [2] the coeficients p, q, p_k are explicitly given while in [3] the uniform convergence of the sequence G_m was proved.

The following denotation will be of use to us:

$$e_0(t) = 1, \quad e_1(t) = t, \quad w(t, c) = |t - c|. \tag{7}$$

3. In this part we shall prove a very simple theorem having large applications in the theory of convex functions. Its proof is based on theorem 1.

Theorem 2 (On the positivity of linear operators). *Let us assume that the operator A: $C[a, b] \to S(D)$ is linear and continuous. Then for every function $t \mapsto f(t)$ the following implication*

$$f \in K[a, b] \Rightarrow Af \geqq 0 \tag{8}$$

is valid if and only if the following three conditions hold:

$$Ae_0 = 0, \tag{9}$$

$$Ae_1 = 0, \tag{10}$$

$$Aw(t, c) \geqq 0 \quad \textit{for every} \quad c \in [a, b]. \tag{11}$$

Proof. (i) Conditions are necessary. Let us assume that the implication (8) holds for any function f and let us prove that the conditions (9), (10) and (11) are valid then.

Since $e_0 \in K[a, b]$ on the basis of the implication (8) it follows that

$$Ae_0 \geqq 0. \tag{12}$$

On the other hand we have $-e_0 \in K[a, b]$ so that we have

$$-Ae_0 = A(-e_0) \geqq 0 \tag{13}$$

on the basis of the linearity of the operator A and on the basis of (8). From (12) and (13) it follows that (9) holds.

Similarly, since $e_1 \in K[a, b]$ and $-e_1 \in K[a, b]$ we see that the following inequalities hold respectively

$$Ae_1 \geqq 0 \tag{14}$$

and

$$-Ae_1 = A(-e_1) \geqq 0 \tag{15}$$

on the basis of linearity of the operator A and on the basis of the implication (8). In virtue of (14) and (15) we see that (10) is valid.

Since $w(t, c) \in K[a. b]$ for every $c \in [a, b]$ on the basis of implication (8) the validity of (11) follows. This proves that the conditions (9), (10) and (11) are necessary.

(ii) Conditions are sufficient. Let us assume that the conditions (9), (10) and (11) are valid and let us prove that the implication holds true, for every function $f \in K[a, b]$.

If $f \in K[a, b]$ then on the basis of the theorem 1, there exist $p, q \in R$, $p_k \geqslant 0$ and $t_k \in [a, b]$, such that the sequence G_m of the form (6) satisfies the conditions

$$\lim_{m \to +\infty} \| G_m - f \| = 0. \tag{16}$$

Since A, by the assumptions of the theorem, is a continuous operator, on the basis of (16) the following relation

$$\lim_{m \to +\infty} \| AG_m - Af \|_1 = 0 \tag{17}$$

is valid for every function $f \in K[a, b]$. On the other hand, since the operator A is linear, in virtue of (6) we have

$$AG_m = pAe_1 + qAe_0 + \sum_{k=0}^{m} p_k Aw(t, t_k). \tag{18}$$

Using now assumptions (9), (10) and (11) it follows that

$$AG_m = \sum_{k=0}^{m} p_k Aw(t, t_k) \geqslant 0 \qquad (m = 1, 2, \ldots) \tag{19}$$

because $p_k \geqq 0$ $(k = 0, 1, \ldots, m)$. On the basis of (17) and (19) we have

$$Af = A(\lim_{m \to +\infty} G_m) = \lim_{m \to +\infty} AG_m \geqq 0$$

for any function $f \in K[a, b]$.

Thereby the theorem 2 is proved.

We shall often, in applications, use the entire family of operators instead of one operator A. Namely, if $\{A_i \mid i \in I\}$ is a family of operators, where I is an arbitrary index-set, a theorem, similar to theorem 2, could be stated for this family.

Theorem 3. *Let us assume that every operator A_i: $C[a, b] \to S(D)$, where I is an arbitrary index-set, is linear and continuous. Then for any function f and for every $i \in I$ the implication*

$$f \in K[a, b] \Rightarrow A_i f \geqq 0 \tag{20}$$

hold if and only if the following conditions are valid

$$A_i e_0 = 0 \tag{21}$$

$$A_i e_1 = 0 \tag{22}$$

$$A_i w(t, c) \geqq 0 \quad \textit{for every} \quad c \in [a, b], \tag{23}$$

for all $i \in I$.

The proof of this theorem is a direct consequence of theorem 2. We shall later, on examples, show that theorem 3 is often more efficient than theorem 2.

As it seems to us, one of the very important consequences of our theorem 2 is the following theorem (see theorem 4). This theorem provides an entirely general principle to the "majorization of vectors" in terms of continuous and linear operators. This theorem contains in a particular case the theorem on the majorization of vectors (see, for example, **[1]** pp. 157—164.).

We will consider two operators A, B: $C[a, b] \to S(D)$. Operator A will be said to majorize the operator B if the following conditions

$$Ae_0 = Be_0 \tag{24}$$

$$Ae_1 = Be_1 \tag{25}$$

$$Aw(t, c) \geqq Bw(t, c) \quad \text{for every} \quad c \in [a, b] \tag{26}$$

are fulfilled, where the functions e_0, e_1 and w are given by (7). The fact that the operator A majorizes the operator B will be denoted by $A \succ B$. The relation $\geqq$ in (26) is defined in the second part of this paper.

Theorem 4 (On the majorization of linear operators). *Let us assume that the operators A, B: $C[a, b] \to S(D)$ are linear and continuous. Then for every function $t \mapsto f(t)$ the implication*

$$f \in K[a, b] \Rightarrow Af \geqq Bf \tag{27}$$

is valid if and only if the operator A majorizes the operator B, i.e. if and only if

$$A \succ B. \tag{28}$$

The proof of this theorem immediately follows from theorem 2 with the fact that the operator $C = A - B$ is linear and continuous if the same properties are possesed by the operators A and B. On the other hand from theorem 4, theorem 2 follows assuming that $B = 0$ in theorem 4. Thus theorems 2 and 4 are equivalent to each other.

In connection with the previous theorems we shall make here some remarks.

REMARKS. 1° For a function f, continuous on the segment $[a, b]$ we will say that it belongs to the class $\bar{K}[a, b]$ (i.e. that it is concave on the segment $[a. b]$) if the function $-f$ belongs to the class $K[a, b]$.

It is easy to verify that the previous theorems continue to be valid if the class K is substituted by the class $\bar{K}$ and if the relation $\geqslant$ is replaced by the relation $\leqslant$ in the mentioned theorems.

2° Analogously to the previous theorems, the necessary and sufficient conditions can be given that for every function f the implication

$$f \in K[a, b] \Rightarrow Af \leqq 0$$

is valid, where A is a linear and continuous operator. It is sufficient to consider the operator $-A$ instead of the operator A. The similar results can be obtained for the class $\bar{K}$.

3° All the operators mentioned in the first part of this paper satisfy the conditions of linearity and continuity in a subspace $S(D)$. In such a way we reach a conclusion that various known theorems of the theory of convex functions could be reduced to the same principle and that on the other hand theorems 2 and 4 enable obtaining of the entirely new results from the theory of convex functions. This will be the topic of several forthcoming notes.

At the end this note, we shall quote another theorem which, analogously to the previous ones, has large applications in the theory of convex functions. Namely, we shall consider the operators A: $C[a, b] \times C[a, b] \rightarrow S(D)$ satisfying the condition

$$A(p_1 u_1 + q_1 v_1, p_2 u_2 + q_2 v_2) = p_1 p_2 A(u_1, u_2) + p_1 q_2 A(u_1, v_2) + p_2 q_1 A(v_1, u_2) + w_1 q_2 A(v_1, v_2),$$

for all real numbers $p_i, q_i \in \mathbf{R}$ $(i = 1, 2)$ and all real functions $u_i, v_i \in C[a, b]$ $(i = 1, 2)$. As it is customary such an operator will be called bilinear. A bilinear operator A is said to be continuous if the operators $Bf = A(f, g)$ and $Cf = A(g, f)$ are continuous on $C[a, b]$, for any function $g \in C[a, b]$. It is verified immediately that the following theorem is valid.

Theorem 5. *Let A: $C[a, b] \times C[a, b] \rightarrow S(D)$ be a bilinear and continuous operator. Then, for every pair (f, g) of the functions the following implication*

$$(f, g) \in K[a, b] \times K[a, b] \Rightarrow A(f, g) \geqq 0 \tag{29}$$

is valid if and only if the following conditions hold

$$A(e_i, e_j) = 0 \quad (i, j = 0, 1) \tag{30}$$

$$A(e_i, w(t, c)) = A(w(t, c), e_i) = 0 \tag{31}$$

for every $c \in [a, b]$ and $i = 0, 1$, and

$$A(w(t, c_1), w(t, c_2)) \geqq 0 \tag{32}$$

for every pair $(c_1, c_2) \in [a, b] \times [a, b]$.

The proof of theorem 5 is very similar to that of theorem 2 and that is why it is omited here.

REMARKS. 4° Theorem 5 is equivalent to theorem 2. Namely, the conditions (30), (31) and (32) can be, by a definite procedure, obtained from the conditions (9), (10) and (11).

5° It is easily seen that theorems 3 and 4 could be transfered to continuous bilinear operators.

6° Theorem 5 can be easily stated even for multilinear operators.

7° Using the well known relation between convex and logarithmically convex functions it is possible to obtain analogous theorems to that obtained in this paper for the class of logarithmically convex functions.

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