UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. Fiz. № 602—№ 633 (1978), 51—52.

# 606. ZEROS OF SUCCESSIVE DERIVATIVES OF A FUNCTION ANALYTIC AT INFINITY

*R. P. Boas*

*Dedicated to Professor D. S. Mitrinović on his seventieth birthday*

If a function $f$ is analytic at 0 (and not a polynomial), its Maclaurin series must have infinitely many nonzero terms, so that an infinite number of its successive derivatives must fail to be zero at 0, and hence must not have zeros in some neighborhood of 0 (the neighborhood depending on the derivative). There are many ways of quantifying this observation; for example, if $f$ is analytic in $|z|<R$ and $z_n$ is the zero of $f^{(n)}$ closest to 0, there is a positive number $G$ such that $|z_n|\geqq RG/n$ for an infinite number of values of $n$. The least upper bound of such numbers $G$ is known as the GONTCHAROFF constant; its precise numerical value is not known, but it was shown by BUCKHOLTZ [1] to be equal to the WHITTAKER constant, which is involved in a similar problem for entire functions and has been studied more extensively.

Here I consider an analogous problem for functions that are analytic at $\infty$. Suppose that $f$ is analytic at $\infty$ and not constant. What can be said about the finite zero of $f$ of largest absolute value? It is not immediately clear that there is anything to say, since the problem is rather different from the original one, principally because the derivatives of $f(z)$ are not simply related to those of $f(1/z)$. Moreover, differentiation does not reduce the influence of the leading terms of a power series in $1/z$. It turns out that if $z_n$ is the finite zero of $f^{(n)}(z)$ of largest absolute value, then $|z_n|$ is at most of order $n$, for all sufficiently large $n$ (not just for a subsequence).

**Theorem.** *Let $f$ be analytic at $\infty$ and for $|z|>R$, and not constant, with $f(\infty)=0$. Let the Laurent series of $f$ (about $\infty$) be $\sum_{n=1}^{+\infty} b_n z^{-n-p}$, where $p$ is a nonnegative integer and $b_1=1$, so that there is a positive $S(S\geqq R$ and $S\geqq 1)$ such that $|b_n|\leqq S^{n-1}$ for $n=1, 2, \ldots$ (since $R=\limsup |b_n|^{1/n}$). Then there is a number $c$, depending only on $p$, such that outside a disk (center at 0) of radius $cS(n+p+1)$ each derivative $f^{(n)}(z)$ of sufficiently large order has no finite zero.*

The example $f(z)=z^{-1}-z^{-2}$, for which $f^{(n)}(z)$ has a zero at $z=n+1$, shows that the radius cannot in general have order greater than $O(n)$.

When $|z|>S$ we have $f(z)=\sum_{j=1}^{+\infty} b_j z^{-j-p}$, $b_1=1$, $|b_j|\leqq S^{j-1}$. Then

$$(-1)^n f^{(n)}(z)=\sum_{j=1}^{+\infty} b_j z^{-j-p-n}(j+p)(j+p+1)\cdots(j+p+n-1)$$

$$=\sum_{j=1}^{+\infty} b_j z^{-j-p-n}\frac{(j+p+n-1)!}{(j+p-1)!}=\sum_{j=1}^{+\infty} b_j z^{-j-p-n}\binom{j+p+n-1}{n} n!.$$

We shall certainly have $f^{(n)}(z)\neq 0$ if

$$|z|^{-n-p-1}(p+n)!/p!>\sum_{j=2}^{+\infty}|b_j z^{-j-p-n}|\binom{j+p+n-1}{n} n!,$$

that is, if

(1) $$1>\frac{p!\,n!}{(p+n)!}\sum_{j=2}^{+\infty}|b_j z^{-j+1}|\binom{j+p+n-1}{n}.$$

With $m=j-1$, (1) reads $1>p!\sum_{m=1}^{+\infty}|b_{m+1}||z|^{-m}\frac{(m+p+n)!}{(m+p)!\,(p+n)!}$, and is implied (since $|b_n|\leqq S^{n-1}$) by

(2) $$1>p!\sum_{m=1}^{+\infty}(S/|z|)^m\binom{m+p+n}{m}.$$

Now if (2) is true for some $|z|$ it is true for any larger $|z|$. Let us take $|z|=cS(n+p+1)$, where $c$ is to be chosen so that (2) will hold. That is, we want to make

$$1>p!\sum_{m=1}^{+\infty}\big(c(n+p+1)\big)^{-m}\binom{m+p+n}{m}=p!\left(\left(1-\frac{1}{c(n+p+1)}\right)^{-(n+p+1)}-1\right).$$

Let $k=n+p+1$. We need

(3) $$1>p!\left(\left(1-\frac{1}{ck}\right)^{-k}-1\right).$$

For sufficiently large $k$, $\big(1-(ck)^{-1}\big)^{-k}$ is arbitrarily close to $e^{1/c}$, and consequently (3) holds if $c$ is chosen large enough so that $p!\,(e^{1/c}-1)<1$, and $k$ is sufficiently large. Hence (1) holds if $c$ is large enough (depending only on $p$; it will suffice to have $c>2p!$), and $n$ is large enough (depending only on $c$, and hence on $p$). It follows that $f^{(n)}(z)\neq 0$ for $|z|>cS(n+p+1)$ when $c$ and $n$ are sufficiently large.

**Added in proof.** While this note was in press, C. L. Prather and J. K. Shaw pointedout that the theorem is a consequence of results of D. V. Widder, Trans. Amer. Math. Soc. **36** (1934), 107—200; see pp. 172—173.

## REFERENCE

1. J D. Buckholtz: *Successive derivatives of analytic functions.* Indian J. Math. **18** (1971), 83—88.

Northwestern University
Evanston, Illinois 60201, U.S.A.