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605. COMPARABLE L_p-NORMS OF SUBADDITIVE FUNCTIONS

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Dedicated to Professor D. S. Mitrinović on the occasion of his seventieth birthday

In a paper published in 1969, F. C. HSIANG [3] proved the following result.

Theorem 1. (a) *Let ψ be positive and monotone increasing and let w be positive on $(0, A)$ where $0<A\leqq+\infty$. Let $p\geqq 1$ and suppose that $w(x)\leqq hx$ for $0<x<A$ and some constant $h>0$. Then there exists an absolute constant $M=M(h, p)>0$ such that for all positive, measurable, subadditive functions φ on $(0, A)$ we have*

$$(1)\qquad \left(\int_0^A \left(\frac{\varphi}{\psi}\right)^p \frac{dx}{w}\right)^{1/p} \leqq M \int_0^A \left(\frac{\varphi}{\psi}\right)\frac{dx}{w}.$$

(b) *Moreover, if ψ is positive and monotone decreasing on $(0, A)$ and there are constants c, k such that $0<c<\frac{1}{2}$, $k>0$, and $\psi(cx)\leqq k\psi(x)$ for $0<x<A$, then the inequality* (1) *is still valid for some $M=M(c, k, h, p)>0$.*

This is not quite the form in which the theorem of [3] was stated, but is what was actually proved. Theorem 1 is a generalization of an earlier theorem of R. P. GOSSELIN [1, Th. 1] who dealt with the special case $w(x)=x$, $\psi(x)=x^{\alpha}$, $\alpha\in\mathbf{R}$. In a later paper [2, p. 258] GOSSELIN noted, in a somewhat different context, that his result remained valid if the L_1-norm appearing on the right side (of the special case) of (1) was replaced by the L_p-norm, where now $0<q<p<\infty$.

It is the purpose of this note to show that Theorem 1 can itself be so extended, and to use this result to obtain a similar comparability result when $w(x)\leqq hx^{\beta}$ $(\beta>1)$ but A is finite.

Theorem 2. *Let ψ, φ, w satisfy the hypotheses of Theorem* 1 *and let $0<q<p<\infty$. In case* (a) *there exists an absolute constant $M=M(h, p, q)>0$ such that*

$$(2)\qquad \left(\int_0^A \left(\frac{\varphi}{\psi}\right)^p \frac{dx}{w}\right)^{1/p} \leqq M \left(\int_0^A \left(\frac{\varphi}{\psi}\right)^q \frac{dx}{w}\right)^{1/q},$$

while in case (b), (2) *remains valid for some $M=M(c, k, h, p, q)>0$.*

Proof. In case (a) we choose any constant c, $0 \leqq c < \frac{1}{2}$, and let $I^q = \int_0^A (\varphi/\psi)^q w^{-1} \mathrm{d}x$. As in [3], let

$$G = \left\{x \in (0, A) : \varphi(x) \leqq \left(h/\log(2(1-c))\right)^{1/q} I \psi(x)\right\},$$

and $E = (0, A) \setminus G$. Then

$$\int_E \frac{\mathrm{d}x}{w} = \int_E \left(\frac{\psi}{\varphi}\right)^q \left(\frac{\varphi}{\psi}\right)^q \frac{\mathrm{d}x}{w} < \frac{\log(2(1-c))}{h I^q} \int_E \left(\frac{\varphi}{\psi}\right)^q \left(\frac{\mathrm{d}x}{w}\right),$$

so

$$\int_E \frac{\mathrm{d}x}{w} < h^{-1} \log(2(1-c)). \tag{3}$$

As in [3], it follows that for every $x \in (0, A)$ there exists $y, z \in (cx, (1-c)x) \cap G$ such that $x = y + z$. Since the proof in [3] contains some misprints which obscure the logic, we provide the proof here. Indeed, if the assertion is false then there exists $x_0 \in (0, A)$ such that for all $y \in (cx_0, (1-c)x_0)$ we have either $y \in E_0 = E \cap (cx_0, (1-c)x_0)$, or $z = x_0 - y \in E_0$, i.e. $y \in x_0 - E_0 = E_1$. Hence

$$(cx_0, (1-c)x_0) = E_0 \cup E_1.$$

Since the sets E_0, E_1 have the same Lebesgue measure, we have

$$(1-2c)x_0 = (1-c)x_0 - cx_0 = |E_0 \cup E_1| \leqq 2|E_0|,$$

or $|E_0| \geqq \left(\frac{1}{2} - c\right)x_0$, so that E_0 occupies at least half of the interval $(cx_0, (1-c)x_0)$. Since x^{-1} has larger values for $x < \frac{1}{2}x_0$ than for $x \geqq \frac{1}{2}x_0$, it therefore follows that

$$\int_{\frac{1}{2}x_0}^{(1-c)x_0} x^{-1}\,\mathrm{d}x \leqq \int_{E_0} x^{-1}\,\mathrm{d}x.$$

Hence by (3),

$$\log(2(1-c)) = \int_{\frac{1}{2}x_0}^{(1-c)x_0} x^{-1}\,\mathrm{d}x \leqq \int_{E_0} x^{-1}\,\mathrm{d}x \leqq h \int_{E_0} \frac{\mathrm{d}x}{w} < \log(2(1-c)),$$

and this contradiction proves the assertion.

Thus for each $x \in (0, A)$ we obtain

$$\varphi(x) = \varphi(y+z) \leqq \varphi(y) + \varphi(z) \leqq (h/\log[2(1-c)])^{1/q} I(\psi(y) + \psi(z)) \tag{4}$$

for appropriate $y,\ z\in G\cap(cx,(1-c)x)$. Since $y,\ z\leqq(1-c)x<x$ while ψ is nondecreasing in case (a), it follows that

$$\varphi(x)\leqq 2\left(h/\log(2(1-c))\right)^{1/q} I\,\psi(x)\quad (0<x<A),$$

so

$$\left(\frac{\varphi}{\psi}\right)^p\frac{1}{w}=\left(\frac{\varphi}{\psi}\right)^{p-q}\left(\frac{\varphi}{\psi}\right)^q\frac{1}{w}\leqq\left(2^q h/\log(2(1-c))\right)^{(p-q)/q} I^{p-q}\left(\frac{\varphi}{\psi}\right)^q\frac{1}{w}.$$

On integrating over $(0, A)$ and taking p^{th} roots, we obtain (2) with

$$M=\left(2^q h/\log(2(1-c))\right)^{1/q-1/p}. \tag{5}$$

It is clear that in this case (a), we may take $c=0$, and this gives the best choice for M in (5).

In case (b) we choose that value of $c\in\left(0,\ \frac{1}{2}\right)$ such that $\psi(cx)\leqq k\psi(x)$ on $(0, A)$. From (4), since ψ is now nonincreasing and $y,\ z\geqq cx$, we obtain

$$\varphi(x)\leqq 2\,k\left(h/\log(2(1-c))\right)^{1/q} I\,\psi(x)\quad (0<x<A).$$

The inequality (2) follows as before, but with

$$M=\left((2\,k)^q h/\log[2(1-c)]\right)^{1/q-1/p}. \tag{6}$$

Corollary. *Let w be positive on $(0, A)$, where $0<A<\infty$, and satisfy $w(x)\leqq hx^\beta$ for some constants $h>0$, $\beta>1$. Let $0<q<p<\infty$ and let ψ, φ satisfy the hypotheses of Theorem* 1. *Then the inequality* (2) *holds with*

$$M=(2^q h A^{\beta-1}/\log 2)^{1/q-1/p}\qquad \textit{in case (a)}, \tag{7a}$$

$$M=((2\,k)^q h A^{\beta-1}/\log[2(1-c)])^{1/q-1/p}\qquad \textit{in case (b)}. \tag{7b}$$

The proof follows at once from the fact that $w(x)\leqq h_1 x$ on $(0, A)$, for $h_1=hA^{\beta-1}$, together with formulas (5) with $c=0$, and (6).

Note that a corresponding result holds for the case $w(x)\leqq hx^\beta\ (\beta>1)$, even if $A=\infty$, provided w is bounded on $(0, A)$. For, if

$$K_w=\sup(w(x):0<x<A)<\infty$$

then again $w(x)\leqq h_2 x$ on $(0, A)$ for $h_2=K_w^{1-(1/\beta)}h^{1/\beta}$. It would be useful to prove a comparability theorem (even for the case $\psi(x)\equiv 1$) without the requirement that w be bounded on $(0, A)$ for $\beta>1$.

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