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604. CONTRIBUTION OF PROFESSOR D. S. MITRINOVIĆ TO DIFFERENTIAL EQUATIONS

— On the occasion of his 70th anniversary —

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1. Introduction

Professor D. S. MITRINOVIĆ published his first paper on differential equations in 1933, and his last, up to now, in 1976, the total number amounting to 99 papers[1]. However, his work in this discipline was not evenly distributed over the period of 43 years — during the nine-year period 1933—1941, Professor MITRINOVIĆ published 47 papers, while in the ten-year period 1967—1976 he published only 6 papers on differential equations.

The first contributions of Professor MITRINOVIĆ were almost entirely devoted to differential equations. In the pre-war period he concentrated his attention mainly to the integration of the equation $(y')^2+y^2=f(x)$, and to the integration of RICCATI's equation. After the end of the war, Professor MITRINOVIĆ began to change his fields of interest, though in the period 1946—1956 he still worked mainly in differential equations. In the post-war period Professor MITRINOVIĆ gave important contributions to the integration of undetermined differential equations, and he also published a series of articles with the aim to form new integrable types of differential equations.

We have therefore classified the papers of Professor MITRINOVIĆ into the following five groups:

1. The equation $(y')^2+y^2=f(x)$;
2. RICCATI's equation;
3. Undetermined equations;
4. New integrable types of differential equations;
5. Other papers.

In further text we shall consider each of those groups independently, stressing the main ideas and results in each of them.

[1] The Bibliography of Professor D. S. MITRINOVIĆ, compiled by R. Ž. ĐORĐEVIĆ and R. R. JANIĆ (see Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. №. 180 (1967), 1—35 and №. 461—№ 497 (1974), 195—206) records 100 papers on differential equations. However, papers listed under № 135 (6) and 150 (9) are identical.

2. The equation $(y')^2+y^2=f(x)$

2.1. *Introduction.* Professor MITRINOVIĆ devoted his first research papers to the integration of the equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+y^2=f(x). \tag{2.1.1}$$

This equation is closely connected with the equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+a_2(x)\,y^2+a_1(x)\,y\frac{\mathrm{d}y}{\mathrm{d}x}+a_0(x)=0. \tag{2.1.2}$$

The equation (2.1.1) is clearly a special case of the equation (2.1.2). It is, however, also the canonical form of the equation (2.1.2). Indeed, if we introduce the substitution

$$y=Y\exp\left(-\int a_1\,\mathrm{d}x\right), \tag{2.1.3}$$

equation (2.1.2) becomes

$$\left(\frac{\mathrm{d}Y}{\mathrm{d}x}\right)^2+\left(a_2(x)-a_1(x)^2\right)Y^2=-a_0(x)\,e^{2\int a_1(x)\,\mathrm{d}x}. \tag{2.1.4}$$

If we now replace the variable x by X, using the formula

$$\frac{\mathrm{d}X}{\mathrm{d}x}=\sqrt{a_2(x)-a_1(x)^2}, \tag{2.1.5}$$

the equation (2.1.3) becomes $\left(\frac{\mathrm{d}Y}{\mathrm{d}X}\right)^2+Y^2=F(X)$, with

$$F(X)=-\frac{a_0(x)}{a_2(x)-a_1(x)^2}\exp 2\int a_1(x)\,\mathrm{d}x,$$

and that is an equation of the form (2.1.1).

When MITRINOVIĆ began his investigations regarding the equation (2.1.1) only three integrable forms of that equation were known:

$$\text{(i)}\quad f(x)=ae^{bx};\qquad \text{(ii)}\quad f(x)=(ax+b)\,e^{\pm 2ix};$$

$$\text{(iii)}\quad f(x)=a\cos\frac{2x}{3}+b\sin\frac{2x}{3},$$

where a and b are arbitrary constants.

2.2. *New integrable forms of* (2.1.2). Taking as a starting point the fact that equation (2.1.1) can be integrated only in the three mentioned cases, MITRINOVIĆ decided to determine (explicitely) as many forms of the function $f(x)$ as possible, which will present integrable cases of the equation (2.1.1). The idea was mainly to reduce the corresponding equation (2.1.2) to known integrable types.

So, for example, in [1] MITRINOVIĆ examined for what values of the coefficients a_0, a_1, a_2 will the equation (2.1.2), solved with respect to y, be a LAGRANGE differential equation

$$y = x A(y') + B(y').$$

Those considerations lead to the conclusion that the equation (2.1.1) will be integrable if f has one of the following forms:

$$f(x) = \frac{\Delta}{k} \exp 2 \int \frac{\lambda t + \mu}{\Delta} \, dt, \quad x = \pm \sqrt{-k} \int \frac{\nu t + \tau}{\Delta} \, dt, \tag{2.2.1}$$

where $\Delta = (\lambda^2 - k\nu^2) t^2 + 2(\lambda\mu - k\nu\tau) t + \mu^2 - k\tau^2$, and λ, μ, ν, k, τ are arbitrary constants;

$$f(x) = (ax + b) e^{\pm 2ix}; \tag{2.2.2}$$

$$f(x) = (ae^{\pm 2i\nu x} + b) e^{\pm 2i(1+\nu)x}, \tag{2.2.3}$$

where a, b, ν are constants and i is the imaginary unit.

REMARK. Although the formulas (2.2.1) which define the function f, are rather complicated, by suitable choices of the arbitrary constants it is possible to form quite simple forms for f. So, for instance, if we take $\nu = 0$, the equalities (2.2.1), after the elimination of t, yield

$$f(x) = (A \cos \lambda x + B \sin \lambda x)^{-2 - \frac{2}{\lambda}},$$

which for $\lambda = -2/3$ reduces to the known form (iii).

Similarly, if we take $\mu = \tau = 0$, we arrive at the form (i).

Since (2.2.2) coincides with (ii), we conclude that formulas (2.2.1), (2.2.2) and (2.2.3) contain all the results which were known up to then.

Developing this idea, MITRINOVIĆ investigated in which cases can the equation (2.1.2) be reduced to an equation of the form

$$y = A(x) y' + B(x) F(y'), \tag{2.2.4}$$

which can be integrated by the method of differentiation.

In the mentioned paper [1] MITRINOVIĆ first showed that the following special case of (2.2.4):

$$y = xy' + cx^n F(y') \qquad (c, n = \text{const}) \tag{2.2.5}$$

is an integrable type, and then proceeded to reduce (2.1.2) to (2.2.5). As a result, two more integrable types of (2.1.1) were found. The function $f(x)$ in those cases is:

$$\begin{cases} f(x) = \dfrac{1}{k} \left((t + \beta t^n)^2 - k\alpha^2 t^{2n} \right) \exp - 2 \displaystyle\int \frac{(t + \beta t^n)}{(t + \beta t^n)^2 - k\alpha^2 t^{2n}} \, dt, \\ x = \pm \alpha \sqrt{-k} \displaystyle\int \frac{t^n}{(t + \beta t^n)^2 - k\alpha^2 t^{2n}} \, dt, \end{cases}$$

$$f(x) = 4 \beta t^n (t + \alpha t^n) e^{\pm 2ix}, \quad x = \pm \frac{i}{2} \int \frac{1}{t + \alpha t^n} \, dt,$$

where k, α, β, n are constants.

A similar idea is applied in [**2**]. In that paper equation (2.1.2) is reduced to the equation

$$xF(y')+yG(y')+(xy'-y)^m H(y)=0 \qquad (m=\text{const}) \tag{2.2.6}$$

which, in turn, after an application of the LEGENDRE transformation becomes a BERNOULLI type equation. MITRINOVIĆ examined the cases when $m=\pm 1$, ± 2, and found two more forms for the function $f(x)$ which give integrable cases of (2.1.1). They are:

$$f(x)=\frac{\mu t^2(t+\lambda)}{\nu^2-(\lambda+2\nu)t}\exp 2\int\frac{\nu-t}{t(t+\lambda)}\,dt, \quad x=\pm\int\frac{\sqrt{(\lambda+2\nu)t-\nu^2}}{t(t+\lambda)}\,dt;$$

$$f(x)=\frac{4(\lambda t^2+\nu)}{(\lambda+\mu)^2t^2+4\mu\nu}\exp(\mu-\nu)\int\frac{t}{\lambda t^2+\nu}\,dt, \quad x=\pm\int\frac{\sqrt{-(\lambda+\mu)^2t^2-4\mu\nu}}{2(\lambda t^2+\nu)}\,dt,$$

where λ, μ, ν are constants.

In his dissertation [3] MITRINOVIĆ extended the methods of [**1**] and [**2**] insomuch as to examine when will (2.1.1) be reduced to an integrable RICCATI type equation. He thus obtained the following integrable forms of (2.1.1):

$$f(x)=\frac{4(\alpha t^2+\gamma)^{1+(\beta-\gamma)/2\alpha}}{(\alpha+\beta)^2t^2+4\beta\gamma}, \quad x=\pm\int\frac{\sqrt{-(\alpha+\beta)^2t^2-4\beta\gamma}}{2(\alpha t^2+\gamma)}\,dt;$$

$$f(x)=-4a\frac{\sqrt{2t-b}}{(t-b)^2}, \quad t=e^{\pm 2ix}\left(e^{\pm 2ix}\mp\sqrt{e^{\pm 4ix}-b}\right);$$

$$f(x)=4a(b-e^{\pm 2ix})^{-1};$$

$$f(x)=\delta\frac{t^{2(\gamma+\varepsilon)/\gamma}(t+\gamma)^{-(\gamma+2\varepsilon)/\gamma}}{\varepsilon^2-(\gamma+2\varepsilon)t}, \quad x=\pm\int\frac{\sqrt{(\gamma+2\varepsilon)t-\varepsilon^2}}{t(t+\gamma)}\,dt,$$

where α, β, γ, δ, ε are constants.

It is also shown that the equation (2.1.1) can be reduced to a RICCATI type equation if $f(x)$ is defined by:

$$f(x)=-\frac{(\beta+\delta t)(t^2+\gamma t+\alpha)}{(\gamma+2\varepsilon)t+\alpha-\varepsilon^2}\exp 2\int\frac{\varepsilon-t}{t^2+\gamma t+\alpha}\,dt,$$

$$x=\pm\int\frac{(\gamma+2\varepsilon)t+\alpha-\varepsilon^2}{t^2+\gamma t+\alpha}\,dt \qquad (\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon=\text{const}),$$

no matter whether the resulting RICCATI's equation is integrable or not.

Finally, in paper [**4**] MITRINOVIĆ reduces the integration of (2.1.1) to the integration of the equation

$$b_0(Y')X^2+2b_1(Y')XY+b_2(Y')Y^2+b_3(Y')=0 \qquad \left(Y'=\frac{dY}{dX}\right). \tag{2.2.7}$$

A number of integrable forms of (2.2.8) is presented, and those are used to prove that (2.1.1) is integrable provided that:

$$f(x)=(A\cos ax+B\sin ax)^{-2-\frac{2}{a}}+(C\cos ax+D\sin ax)^{-2-\frac{2}{a}},$$

where a, A, B, C, D are constants such that $AC+BD=0$. Some simple special cases of (2.2.8) are:

$$f(x)=A\sin x,\ f(x)=A\cos x,\ f(x)=(A\cos ax+B\sin ax)^{-2-\frac{2}{a}};$$

$$\begin{cases} f(x)=\dfrac{(a_3-a_7\,t)\,(a_0+3\,a_1\,t+3\,a_2\,t^2-a_6\,t^3)^{1/3}}{(a_1\,a_6+a_2^2)\,t^2+(a_0\,a_6+a_1\,a_2)\,t+a_1^2-a_0\,a_2} \\ x=\pm\displaystyle\int\frac{(a_0\,a_2-a_1^2-(a_0\,a_6+a_1\,a_2)\,t-(a_1\,a_6+a_2^2)\,t^2)^{1/2}}{a_0+3\,a_1\,t+3\,a_2\,t^2-a_6\,t^3}\,\mathrm{d}t \end{cases};$$

$$f(x)=A+B\log\operatorname{tg}(x+a);\qquad f(x)=A+B\log\operatorname{cotg}(x+a);$$

$$f(x)=A+B\operatorname{arctg}(C+D\operatorname{cotg}x);\quad f(x)=A+B\log\frac{1-\operatorname{cotg}x}{1+\operatorname{cotg}x};$$

$f(x)$ is any root of the equation $\alpha f+\beta f^k=e^{\pm 2ix}$.

On the other hand, equation (2.1.1) can be reduced to a RICCATI type equations if:

$$\begin{cases} f(x)=\dfrac{(a_3-a_5\,t)\,a_0+(2\,a_1-a_4)\,t+a_2\,t^2}{a_1^2-a_0a_2+a_2\,a_4\,t}\exp-2\displaystyle\int\frac{a_1+a_2\,t}{a_0+(2\,a_1-a_4)\,t+a_2\,t^2}\,\mathrm{d}t \\ x=\pm\displaystyle\int\frac{\sqrt{a_0\,a_2-a_1^2-a_2\,a_4\,t}}{a_0+(2\,a_1-a_4)\,t+a_2\,t^2}\,\mathrm{d}t \end{cases};$$

or if $f(x)$ is any solution of the equation

$$\alpha f^k+\beta f+\gamma f^{2-k}=e^{\pm 2ix}.$$

Special attention is paid to the resulting RICCATI's equations which are integrable, since such equations lead to new integrable types of (2.1.1).

In paper **[5]** it is shown that (2.1.1) can be solved using HERMITE's polynomials, provided that

$$f(x)=\pm i\operatorname{tg}x+(2n+1)\quad\text{or}\quad f(x)=\pm i\operatorname{cotg}x+(2n+1),$$

where i is the imaginary unit, and $n\in\mathbf{Z}$.

We also mention paper **[6]** which contains an explicit solution of the equation $(y')^2+y^2=A\sin x\,(A=\text{const})$, although this was already done in **[4]**.

2.3. *Connection with other equations.* An other approach used by MITRINOVIĆ in relation with the equation (2.1.1) was to reduce that equation to other equations which are also not integrable in general, but which are more familiar than the equation (2.1.1).

For example, in paper [7] MITRINOVIĆ found a connection between ABEL's equation

$$\frac{\mathrm{d}u}{\mathrm{d}x}=(1+u^2)\left(1-\frac{1}{2}\,\frac{f'(x)}{f(x)}\,u\right)$$

and the equation (2.1.1), concluding that each case of integrability of one of those two equations leads to a corresponding case of integrability for the other.

Some very general results were obtained in [8]. In that paper MITRINOVIĆ proved, among other things, the following theorems:

1° If one particular solution of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{a_0(x)\,y^2+a_1(x)\,y+a_2(x)}{y+b(x)} \tag{2.3.1}$$

is known, than that equation can be reduced to the equation (2.1.1).

2° If two particular solutions of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}=a_0(x)\,y^3+a_1(x)\,y^2+a_2(x)\,y+a_3(x) \tag{2.3.2}$$

are known, then that equation can be reduced to the equation (2.1.1).

3° It is possible to form all integrable cases of the equation (2.1.1).

4° Each case of integrability for any one of the equations: (2.3.1), (2.3.2), $y'=a_0(x)+a_1(x)\,y^{-1}$, or

$$y'=\frac{a_0(x)\,y^3+a_1(x)\,y^2+a_2(x)\,y+a_3(x)}{y+b(x)}$$

leads to a case of integrability for the equation (2.1.1).

The first two results of Theorem 4 are clearly consequences of Theorems 1 and 2. Theorem 3 is proved in the following way. It is first shown that the exterior balistic equation

$$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1-u^2}{u+g(x)} \tag{2.3.3}$$

can be transformed into equation (2.1.1). On the other hand, J. DRACH applied his method of logical integration to (2.3.3) and showed how one can form all the functions $g(x)$ such that (2.3.3) is integrable. Hence, it follows that it is possible to form all the functions $f(x)$ which ensure the integrability of (2.1.1). This constatation completely solves the question of integrability of the equation (2.1.1).

Papers [9], [10] and [11] are short extracts from the paper [8].

2.4. *A more general equation.* MITRINOVIĆ devoted a number of papers to the integration of the equation

$$A(x)(y')^2+2\,B(x)\,yy'+C(x)\,y^2+2\,D(x)\,y'+2\,E(x)\,y+F(x)=0. \tag{2.4.1}$$

We first note the following:

(i) If

$$\begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} = 0,$$

equation (2.4.1) decomposes into two linear equations;

(ii) If $A=0$, $B^2-AC=0$, (2.4.1) is a RICCATI type equation.

(iii) If $A=0$, $B^2-AC\neq 0$, (2.4.1) is an ABEL type equation.

MITRINOVIĆ in paper [**12**] investigated the remaining two cases, namely: (iv) $A\neq 0$, $B^2-AC\neq 0$, and (v) $A\neq 0$, $B^2-AC=0$. He showed that in the case (iv) equation (2.4.1) is equivalent to an equation of the form

$$\frac{du}{dx}=\frac{M(x)\,u^3+N(x)\,u^2+P(x)\,u}{u^2+Q(x)}, \tag{2.4.2}$$

which means that integrability of (2.4.2) implies integrability of (2.4.1).

In case (v) MITRINOVIĆ proved that the equation (2.4.1) can be reduced to an ABEL type equation

$$\frac{du}{dx}=\frac{M(x)\,u^2+N(x)\,u+P(x)}{u+Q(x)}. \tag{2.4.3}$$

On the other hand integrability of any one of the equations

$$\frac{du}{dx}=A(u)+xB(u), \quad \text{or} \quad \frac{du}{dx}=A(u)+x^{-1}B(u) \tag{2.4.4}$$

implies the integrability of ABEL's equation (2.4.3), which connects the equation (2.4.1) in case (v) with the equations (2.4.4).

In paper [**8**] MITRINOVIĆ obtained the conditions which allow the equation (2.4.1) to be reduced to the equation (2.1.2). Those conditions are:

$$A\neq 0, \ B^2-AC\neq 0, \ \left(\frac{AE-BD}{B^2-AC}\right)'=\frac{CD-BE}{B^2-AC}.$$

2.5. *Geometrical problems in connection with* (2.1.1) *and* (2.1.2). MITRINOVIĆ published a number of papers in which various problems of differential geometry are reduced to the integration of equations (2.1.1) and (2.1.2). We shall briefly mention some of the results he obtained.

The following problem was solved in [**13**]: Determine all the surfaces $z=F(x, y)$ with the property that (2.1.2) is their differential equation of asymptotic lines. The solution is rather complicated, but it contains some simple special cases. For example, surfaces of the form

$$z=f(x)\,y^2+ky+g(x) \qquad (k=\text{const}) \tag{2.5.1}$$

have the required property. The special surface (2.5.1) was again treated in [**14**].

The same problem was considered in [15], [16], but the surfaces are taken in the form

$$z+vy=\sum_{k=0}^{n} f_k(x)\,v^{n-k},\qquad y=\sum_{k=0}^{n-1}(n-k)\,f_k(x)\,v^{n-k-1},$$

and it is requested that the differential equation of the asymptotic lines is $(\mathrm{d}v/\mathrm{d}x)^2=\big(A(x)\,v+B(x)\big)^2$ in [15], and $(\mathrm{d}v/\mathrm{d}x)^2=A(x)\,v^2+B(x)\,v+C(x)$ in [16].

In paper [17] it is shown that the determination of the curvature lines of the surface $z=f(x)\,y+g(x)$, where g satisfies a certain condition, can be reduced to the integration of a special equation of type (2.4.1) which is integrable. Similarly, in paper [18] it is shown that the determination of the asymptotic lines of the surface $z=\sum_{k=0}^{3} f_k(x)\,y^k$, under rather strict restrictions on the functions f_0, f_1, f_2, f_3 can be reduced to the integration of (2.4.1), and then, under some additional conditions to the integration of (2.1.1).

Paper [19] contains short summaries of the results from [8], [15] and [16]. Finaly, in [20] BELTRAMI's problem is reduced to the integration of the equation (2.1.2) and (2.1.1).

1.6. *An other result for the equation* (2.1.1). In connection with the equation (2.1.1) MITRINOVIĆ [21] proved the following result: Suppose that the general solution of the equation (2.1.1) has the form $y=F(x, C)$, where C is the integration constant. The function F will be a rational function with respect to C if and only if $f(x)=\text{const}$. The corresponding result for the equation (2.1.2) is also given.

1.7. *Comment.* Professor MITRINOVIĆ devoted a considerable number of papers to a very specific problem: integration of the equation (2.1.1). This problem, though easy to formulate, is very complicated. By using various methods Professor MITRINOVIĆ developed a large number of integrable equations of type (2.1.1), and he also showed that the integration of that equation can be reduced to the integration of some well known differential equations. Besides, he obtained, as by-products, various new types of integrable differential equations.

It is interesting to note that KAMKE in his famous collection [103] did not include MITRINOVIĆ's results on the integrable forms of (2.1.1). The only references to MITRINOVIĆ's research connected with that equation are given in KAMKE's equation 1.395, where the transformation (2.1.3)—(2.1.5) is reproduced (though this transformation is not really due to MITRINOVIĆ), and in equation 1.461, which is concerned with the equation (2.4.1). On the other hand, equations (2.2.5) and (2.2.6), which MITRINOVIĆ obtained as by-products, are recorded by KAMKE as new integrable types (the equations 1.571 and 1.572).

The conslusion that it is possible to construct all the functions f such that the equation (2.1.1) is integrable presents a natural end of MITRINOVIĆ's investigations regarding that equation. On the other hand, the problem is not completely solved; namely, the effective construction of all the integrable cases, applying DRACH's logical integration, remained unfinished. It is possible that the beginning of the war prevented Professor MITRINOVIĆ in giving the complete answer to the problem he set himself.

3. Riccati's equation

3.1. *An integrability criterion.* Professor MITRINOVIĆ [22] proved the following result regarding the RICCATI equation

$$y' = A(x)y^2 + B(x)y + C(x): \tag{3.1.1}$$

(i) If there exists a function F such that

$$A\left(\frac{F-B}{2A}\right)^2 - F\frac{F-B}{2A} + \left(\frac{F-B}{2A}\right)' = C, \tag{3.1.2}$$

then the equation (3.1.1) can be integrated by quadratures;

(ii) If the function F from (3.1.2) has the form G'/G, then the general solution of (3.1.1) can be obtained by means of one quadrature;

(iii) If the function F from (3.1.2) has the form $(H''/H)-(A'/A)$, then the general solution of (3.1.1) can be obtained without quadratures.

The above theorem contains as special cases some known results in connection with RICCATI's equation.

3.2. *Formation of integrable Riccati's equation.* The starting point for this line of research was a theorem of DARBOUX, which we shall quote in full.

Darboux's theorem. Suppose that the differential equation

$$y'' = (f(x)+h)y \qquad (h = \text{const}) \tag{3.2.1}$$

is integrable for all values of h. Then there exists an infinity of equations of the same type which are also integrable for all values of h.

Inspired by this result of DARBOUX and a result of M. PETROVIĆ for the RICCATI equation, MITRINOVIĆ [23] solved the following two problems:

PROBLEM 1. Starting with a given integrable RICCATI's equation

$$y' + y^2 = F(x),$$

form an infinite sequence of other RICCATI's equations of the same form, which are also integrable.

PROBLEM 2. Starting with a given RICCATI's equation

$$y' + y^2 = f(x, h),$$

supposed integrable for all values of h, form an infinite sequence of other RICCATI's equations which shall also be integrable for all values of h.

The main result of paper [23] reads:

If the equation

$$y' + y^2 = F(x) \tag{3.2.2}$$

is integrable, then the equations

$$y_k' + y_k^2 = F_k(x) \qquad (k = 1, 2, \dots) \tag{3.2.3}$$

are also integrable, where $F_0=F$,

$$F_k=F_{k-1}+\frac{Q_k''-F_{k-1}'}{Q_k}+\frac{3}{4}\Big(\big(\log(F_{k-1}-Q_k'-Q_k^2)\big)'\Big)^2$$
$$+\frac{F_{k-1}-Q_k'}{Q_k}\big(\log(F_{k-1}-Q_k'-Q_k^2)\big)'-\frac{1}{2}\,\frac{(F_{k-1}-Q_k'-Q_k^2)''}{F_{k-1}-Q_k'-Q_k^2},$$

and $Q_1,\ldots,Q_k$ are arbitrary functions. Moreover, the integral of the equation (3.2.3) is given by the recurrent formulas:

$$y_k=\frac{(F_{k-1}-Q_k'-Q_k^2)\,y_{k-1}}{Q_k\,(y_{k-1}-Q_k)}$$
$$+\frac{Q_k\,Q_k''-Q_k\,F_{k-1}'+2\,Q_k^2\,F_{k-1}-2\,Q_k'^2+4\,Q_k'\,F_{k-1}-2\,F_{k-1}^2}{2\,Q_k\,(F_{k-1}-Q_k'-Q_k^2)},$$

where $y_0=y$ is the integral of the equation (3.2.2).

The idea is quite straightforward. Namely, the RICCATI equation (3.2.2) after the substitution

$$y=\frac{Q(x)\,z}{z+1} \tag{3.2.4}$$

becomes the RICCATI equation

$$z'=A(x)\,z^2+B(x)\,z+C(x), \tag{3.2.5}$$

with

$$A=Q^{-1}(F-Q'-Q^2),\quad B=Q^{-1}(2F-Q'),\quad C=Q^{-1}F.$$

If we introduce the following substitution

$$z=-\frac{1}{A}\,y_1-\frac{1}{2}\Big(\frac{A'}{A^2}+\frac{B}{A}\Big),$$

the equation (3.2.5) takes the canonical form

$$y_1'+y_1^2=F_1(x), \tag{3.2.7}$$

where F_1 depends on A, B, C, or in virtue of (3.2.6) on F and Q. In fact, it is readily shown that

$$F_1=F+\frac{Q''-F'}{Q}+\frac{3}{4}\Big(\big(\log(F-Q'-Q^2)\big)'\Big)^2$$
$$+\frac{F-Q'}{Q}\big(\log(F-Q'-Q^2)\big)'-\frac{1}{2}\,\frac{(F-Q'-Q^2)''}{F-Q'-Q^2}.$$

This procedure can clearly be now applied to the equation (3.2.7) to yield a new RICCATI equation

$$y_2'+y_2^2=F_2(x),$$

where F_2 depends on F_1 and Q_1, etc.

It should be noted that the function F_k in (3.2.3) contains k arbitrary functions, which makes this result rather general.

Starting from special forms of the function F which ensure the integrability of the equation (3.2.2), we can form various particular results. So, for instance, if we put $F(x)=f(x)+h-2Q(x)$, where Q is a solution of the equation

$$Q'+Q^2=f(x)+h_1,$$

we arrive at a result, which after an application of the standard transformations which reduce RICCATI's equations to linear second order equations, becomes exactly DARBOUX's theorem.

Among other special cases of this result there are many theorems which were known, and many theorems which were published later. MITRINOVIĆ returned to his theorem for the RICCATI equation and in papers **[24]**, **[25]** announced that he will systematically expose all the results obtained after the publication of paper **[23]**, but which are special cases of his theorem. So far this work has remained unfinished.

In connection with the Problem 2, MITRINOVIĆ **[23]** gave two solutions in the form of the following theorems:

If the equation

$$y'+y^2=hf(x)$$

is integrable for all values of h, so is the equation

$$y'+y^2=f(x)+h.$$

If the equation

$$y'+y^2=f(x)+h^2$$

is integrable for all values of h, so is the equation

$$y'+y^2=f(x)+\frac{3}{4}\left(\frac{f'}{f}\right)^2-\frac{1}{2}\frac{f''}{f}+h\frac{f'}{f}+h^2.$$

At the end of paper **[23]** MITRINOVIĆ indicated that he will, in an other paper, consider some other forms of the function $f(x, h)$ from Problem 2, but he never did that.

Papers **[26]** and **[27]** contain short summaries from **[23]**.

3.3. *Riccati's equations invariant under the transformation* (3.2.4). Transformations under which the RICCATI's equation is invariant were considered in papers **[28]**, **[29]** and **[30]**. In **[28]** it is shown that certain results of POMPEIU can be obtained as consequences of the fact that RICCATI's equation

$$y'=p(x)y^2+q(x)y+r(x) \tag{3.3.1}$$

after the transformation $y=Q(x)v+R(x)$ takes the form

$$v'=p_1(x)v^2+q_1(x)v+r_1(x), \tag{3.3.2}$$

where $p_1=Qp$, $q_1=2Rp+q-Q^{-1}Q$, $r_1=Q^{-1}(R^2p+Rq+r-R')$, and that the equations (3.3.1) and (3.3.2) are therefore equivalent with respect to integrability.

Note **[29]** is a short summary of **[30]**, and so we shall give only the results from **[30]**. The transformation (3.2.4) is again considered and the following problem is solved:

Determine all RICCATI's equations of the form (3.2.2) which remain invariant under the transformation (3.2.4), i.e. which are transformed by (3.2.4) into the equation $z'+z^2=F(x)$.

The following theorem is proved:

The equation

$$y'+y^2=\frac{3}{4}\left(\frac{G''}{G'}\right)^2-\frac{1}{2}\frac{G'''}{G'}+a\left(\frac{G'}{G}\right)^2 \qquad (a=\text{const}\neq 0) \tag{3.3.3}$$

is invariant under the transformation

$$y=-\frac{1}{2}\frac{G''}{G'}+a\left(\frac{G'}{G}\right)^2\left(z+\frac{1}{2}\frac{G''}{G'}-\frac{G'}{G}\right)^{-1}, \tag{3.3.4}$$

where G is a nonconstant arbitrary differentiable function.

The equation (3.3.3) is clearly integrable. One particular solution of that equation is obtained from the algebraic system consisting of (3.3.4) and the equation $y=z$.

3.4. *Comment.* Professor MITRINOVIĆ did not devote many papers to RICCATI's equation, but he obtained some very general results, which were soon noted in literature. So, for example, KAMKE in the theoretical part of his book **[103]** brings MITRINOVIĆ's result which was exposed here in 3.1. Results from paper **[23]** were given considerable space in BUHL's textbook **[106]**, and they are also mentioned in the recently published monograph **[107]**.

It is interesting to note that in papers **[23]**, **[25]** and **[30]** Professor MITRINOVIĆ announced that he will continue his researches though he never did that. His abandonment of the problems treated in **[23]** can be explained by the outbreak of the war, while in the case of the problems from **[30]** and especially **[25]**, the reason certainly lies in the fact that Professor MITRINOVIĆ at that time completely changed his field of interest.

4. Undetermined differential equations

4.1. *Introduction.* An undetermined differential equation is an equation which contains more than one unknown function. Professor MITRINOVIĆ devoted a number of papers to such equations, especially those of the form

$$F(x, y, y', y'', z, z', z'')=0. \tag{4.1.1}$$

The equation (4.1.1) is completely solved if all pairs (y, z) which identically satisfy it are found. Naturally, that is possible only in a very limited number of cases. However, having in mind that equations of the form (4.1.1) appear very often in applied sciences, it is useful to determine as many particular solutions of such equations as possible.

Professor MITRINOVIĆ introduced three different methods for solving undetermined equations, which enabled him to find some particular solutions of those equations (sometimes an infinite amount of such solutions), and in certain cases to find all the solutions of the equation in question.

The following undetermined equations were considered in papers [31]—[41]:

(4.1.2) $$zy'' - kyz'' = 0 \qquad (k = \text{const})$$

(4.1.3) $$zy'' - yz'' = hyz \qquad (h = \text{const})$$

(4.1.4) $$zy'' - f(x)\, yz'' = g(x)\, yz \qquad (f(x) \neq 0)$$

(4.1.5) $$z^2 y'' = hyzz'' + ky + lyz^2 \qquad (h.\ k,\ l = \text{const})$$

(4.1.6) $$(y'' - 2ay')\, z - yz'' = 0 \qquad (a = \text{const})$$

(4.1.7) $$(z - c)\,(y'' - a^2 y) - yz'' + A\,(y^{(4)} - 2a^2 y'' + a^4 y) = 0$$
$$(a,\ c,\ A = \text{const}).$$

I especially want to underline that the equations (4.1.2)—(4.1.7) are not "artificial". Equation (4.1.2) is the equation to which TRUESDELL and NEMÉNYI reduced the general equilibrium problem in the membrane theory of shells of revolution. The equation (4.1.3) and the equation

(4.1.8) $$z^2 y'' = yzz'' + ky + lyz^2 \qquad (k,\ l = \text{const})$$

appear in a general problem of hydrodinamics which was considered by HÖILAND, PALM, ELIASSEN and RIIS. The equation (4.1.4) is a generalisation of the equations (4.1.2) and (4.1.3),while (4.1.5) is a generalisation of (4.1.3) and (4.1.8). The equation (4.1.6) appears in a technical problem studied by GRAN OLSSON, and finally the equation (4.1.7), which bears the name SOMMERFIELD's equation, was used by TOLLMIEN in connection with a problem of hydrodinamics.

4.2. *The connection with a problem of Darboux-Drach.* The first method used by MITRINOVIĆ consists in replacing the starting equation (4.1.1) by an equivalent system of the form

(4.2.1) $$y'' = \big(F(x) + h\big)\, y,$$

(4.2.2) $$z'' = F(x)\, z,$$

where F is an arbitrary function.

This is useful for the following reason. DARBOUX (see 3.2) constructed an infinite sequence of equations of the form (4.2.1) which are integrable for all h. Moreover, DRACH determined *all* the functions F such that the equation (4.2.1) is integrable for all h. On the other hand, it is clear that each particular choice of such a function F leads to a particular solution of the equation (4.1.1). Hence, this method gives rise to an infinite sequence of the solutions of the equation (4.1.1).

MITRINOVIĆ applied this method to the equations (4.1.2), (4.1.3) and (4.1.7).

The equation (4.1.3) is equivalent to the system (4.2.1)—(4.2.2) and hence it is possible to find an infinite sequence of solutions of that equation.

The equation (4.1.2) is equivalent to the system

(4.2.3) $$y'' = kF(x)\, y, \quad z'' = F(x)\, z,$$

where F is an arbitrary function. The first equation of (4.2.3) can be, by means of certain transformations, reduced to the form $u'' = \big(G(v) + h\big)\, u$, i.e. to the form (4.2.1), which means that the same method can be applied.

The equation (4.1.7), after the transformation $z=v+c$ becomes

$$v''-\frac{y''-a^2y}{y}v=\frac{A}{y}\left((y''-a^2y)''-a^2(y''-a^2y)\right)$$

and this equation is equivalent to the system

$$y''-(F(x)+a^2)y=0, \tag{4.2.4}$$

$$v''-F(x)v=A\left(F''+F^2+2\frac{y'}{y}F'\right), \tag{4.2.5}$$

where F is an arbitrary function.

Instead of the system (4.2.4)—(4.2.5) we may consider the system which consists of the equation (4.2.4) and the equation

$$v''-F(x)v=0, \tag{4.2.6}$$

which is again a system of the type (4.2.1)—(4.2.2), and hence it is possible to construct an infinite sequence of solutions of that system. For each solution of the system (4.2.4)—(4.2.6) we may find the corresponding solution of the system (4.2.4)—(4.2.5), for example by means of the variation of parameters.

REMARK. This method is very specific, since it requests that the given equation can be replaced by a fixed system of the type (4.2.1)—(4.2.2), or by a system which can be reduced to that system. It might be of interest to try some generalisations of this method, in the sense that the given equation (4.1.1) is replaced by an equivalent system, of the form

$$G(x,y,y',y'',F(x)+h)=0,\quad G(x,z,z',z'',F(x))=0, \tag{4.2.7}$$

where F is an arbitrary function. However, in that case it is necessary to know instances of the equation (4.2.7) which are integrable for all h.

4.3. *Reduction to first order equations.* If we put $y=T(z)$ into (4.1.1), where T is an arbitrary twice differentiable function, we arrive at the equation

$$F\left(x,T(z),\ T'(z)z',\ T''(z)z'^2+T'(z)z'',\ z,\ z',\ z''\right)=0. \tag{4.3.1}$$

If the equation (4.3.1) can be solved for arbitrary T, then it is possible to solve the given equation (4.1.1). In particular, if (4.1.1) does not contain x, i.e. if $\partial F/\partial x=0$, then the equation (4.3.1) after the standard transformation $z'=p$, $z''=p\,(\mathrm{d}p/\mathrm{d}z)$ becomes

$$F\left(T(z),\ T'(z)p,\ T''(z)p^2+T'(z)p\frac{\mathrm{d}p}{\mathrm{d}z},\ z,\ p,\ p\frac{\mathrm{d}p}{\mathrm{d}z}\right)=0, \tag{4.3.2}$$

and that is a first order equation.

MITRINOVIĆ applied this method to the equations (4.1.2), (4.1.3). (4.1.5) and (4.1.6). The corresponding equation (4.3.2) reads:

(i) for the equation (4.1.2):

$$\left(\frac{T'(z)}{T(z)}+\frac{k}{z}\right)p\frac{\mathrm{d}p}{\mathrm{d}z}+\frac{T''(z)}{T(z)}p^2=0;$$

(ii) for the equation (4.1.3):

$$\left(\frac{T'(z)}{T(z)}-\frac{1}{z}\right)\frac{\mathrm{d}p}{\mathrm{d}z}+\frac{T''(z)}{T(z)}p=\frac{h}{p}; \tag{4.3.3}$$

(iii) for the equation (4.1.5):

$$\left(\frac{T'(z)}{T(z)}-\frac{h_3}{z}\right)\frac{\mathrm{d}p}{\mathrm{d}z}+\frac{T''(z)}{T(z)}p=\left(\frac{h_2}{z^2}+h_1\right)\frac{1}{p};$$

(iv) for the equation (4.1.6):

$$(zT'(z)-T(z))\frac{\mathrm{d}p}{\mathrm{d}z}+zT''(z)p=2azT'(z).$$

All those equations can be integrated by quadratures. Indeed the first equation has separated variables, the second and the third are BERNOULLI's equations, and the last one is linear.

In order to give a better illustration of this method, we shall completely solve one of them, for example the equation (4.3.3).

The equation (4.3.3) after the substitution $p=\sqrt{u}$ becomes

$$\frac{\mathrm{d}u}{\mathrm{d}z}+\frac{2zT''(z)}{zT'(z)-T(z)}u=\frac{2hzT(z)}{zT'(z)-T(z)}$$

which implies

$$u=\frac{C+2h\int zT(z)\,(zT'(z)-T(z))\,\mathrm{d}z}{(zT'(z)+T(z))^2} \qquad (C \text{ arbitrary constant})$$

wherefrom we get

$$(4.3.4) \qquad x+D=\frac{zT'(z)-T(z)}{\sqrt{C+2h\int zT(z)\,(zT'(z)-T(z))\,\mathrm{d}z}}\,\mathrm{d}z \qquad (D \text{ arbitrary constant}).$$

Hence, the solution of (4.1.3) is obtained in the following way:

(i) The function y is defined by (4.3.4) and $y=T(z)$, where T is an arbitrary differentiable function, C and D are arbitrary constants, and z is a parameter which should be eliminated between those two equalities;

(ii) When y is determined, the function z is defined by $y=T(z)$.

We therefore see that the solution of (4.1.3) is rather complicated, and it cannot be obtained in the explicit form. However, for particular choices of the arbitrary function T, we may arrive at simple explicit solutions of (4.1.3). For instance, if we set $T(z)=z^2$, we find the following solutions of (4.1.3):

$$y=C\left(\mathrm{ch}\,(\sqrt{3h}\,x+\alpha)\right)^{2/3}, \qquad z=D\left(\mathrm{sh}\,(\sqrt{3h}\,x+\alpha)\right)^{1/3} \qquad (h>0);$$

$$y=C\left(\cos\,(\sqrt{-3h}\,x+\alpha)\right)^{2/3}, \qquad z=D\left(\sin\,(\sqrt{-3h}\,x+\alpha)\right)^{1/3} \qquad (h<0).$$

REMARK. This method, as opposed to the method of section 4.2, is much more general. The transformation $y=T(z)$ can always be applied, and it will always lead to a first order equation, under the condition that the starting equation does not contain the independent variable. Of course, it remains to be seen whether this first order equation will be integrable or not. Besides, even in the case when the equation can be integrated, the solution can be very complicated in form.

REMARK. MITRINOVIĆ suggested certain generalisations of this method. In particular in paper [42] he considered the third order equation

$$\frac{y''}{y}-\left(\frac{z'''}{z''}+\frac{z'}{z}\right)\frac{y'}{y}+n^2\frac{z''}{z}=0 \qquad (n\in\mathbf{N}) \tag{4.3.5}$$

to which TRUESDELL in his dissertation reduced an important problem from the theory of elasticity. MITRINOVIĆ obtained a solution of the equation (4.3.5) which contains one arbitrary function and three arbitrary constants, by introducing the substitution $y=T(z')$. In the same paper he also mentioned other possible generalisations.

REMARK. The method of this section was also applied by MITRINOVIĆ [43] to a problem in connection with PETROVIĆ's operator Δ_k defined by $\Delta_k f(x)=f^{(k)}(x)/f(x)$.

4.4. *Lowering the order of the equation.* In certain cases the substitution $y=e^{\int u\,dx}$, $z=e^{\int v\,dx}$ can result in lowering the order of the equation (4.1.1). Indeed, applying that substitution to (4.1.1) we get

$$F\left(x,\ e^{\int u\,dx},\ ue^{\int u\,dx},\ (u'+u^2)\,e^{\int u\,dx},\ e^{\int v\,dx},\ ve^{\int v\,dx},\ (v'+v^2)\,e^{\int v\,dx}\right)=0, \tag{4.4.1}$$

and hence, the function F should satisfy the functional equation

$$\begin{aligned}F(x_1,\ \alpha x_2,\ \alpha x_3,\ \alpha x_4,\ \beta x_5,\ \beta x_6,\ \beta x_7)\\ =f(\alpha)\,g(\beta)\,F(x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6,\ x_7)\end{aligned} \tag{4.4.2}$$

if (4.4.1) is to be a first order equation. It can be proved that all solutions of the equation (4.4.2) are given by

$$F(x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6,\ x_7)=G\left(x_1,\ \frac{x_3}{x_2},\ \frac{x_4}{x_2},\ \frac{x_6}{x_5},\ \frac{x_7}{x_5}\right),\quad f(x)=x^r,\quad g(x)=x^s,$$

where G is an arbitrary function and r, s are arbitrary constants.

Therefore, this method can be applied to equations of the form

$$G\left(x,\ \frac{y'}{y},\ \frac{y''}{y},\ \frac{z'}{z},\ \frac{z''}{z}\right)=0. \tag{4.4.3}$$

In that case we arrive at the following first order equation

$$G(x,\ u,\ u'+u^2,\ v,\ v'+v^2)=0,$$

i.e. to an equation of the form

$$H(x,\ u,\ u',\ v,\ v')=0. \tag{4.4.4}$$

In other words, the integration of (4.4.3) is reduced to the integration of (4.4.4).

Regarding the integration of (4.4.4) MITRINOVIĆ suggested the following method. Suppose that (4.4.4) can be written in the form

$$\frac{d}{dx}\left(M(x,\ u,\ v)\right)+N(x,\ u,\ v)=0. \tag{4.4.5}$$

Then the functions u and v can be determined from the *algebraic* system of equations

$$M(x, u, v) = \Phi(x), \quad N(x, u, v) = -\Phi'(x),$$

where Φ is an arbitrary differentiable function.

MITRINOVIĆ applied this method to the integration of equations (4.1.2), (4.1.3), (4.1.4) and (4.1.6). For those equations the corresponding equation (4.4.4) reads:

(i) for the equation (4.1.2):

$$u' + u^2 + k(v' + v^2) = 0; \tag{4.4.6}$$

(ii) for the equation (4.1.3):

$$u' + u^2 - v' - v^2 = h; \tag{4.4.7}$$

(iii) for the equation (4.1.4):

$$u' + u^2 - f(x)\,(v' + v^2) = g(x); \tag{4.4.8}$$

(iv) for the equation (4.1.6):

$$u' + u^2 - 2au - v' - v^2 = 0, \tag{4.4.9}$$

and all those equations are of the form (4.4.5). Indeed, the equation (4.4.6) is equivalent to $(u + kv)' + (u^2 + v^2) = 0$; the equation (4.4.7) to $(u - v)' + (u^2 - v^2 - h) = 0$, the equation (4.4.8) to $(u - f(x)\,v)' + (u^2 - f(x)\,v^2 + f(x)\,v - g(x)) = 0$, and the equation (4.4.9) to the equation

$$(u - v)' + (u^2 - 2au - v^2) = 0. \tag{4.4.10}$$

We shall completely solve the equation (4.4.9), i.e. (4.4.10). From the system

$$u - v = \Phi(x), \quad u^2 - 2au - v^2 = -\Phi'(x),$$

we find

$$u = \frac{\Phi' - \Phi^2}{2(a - \Phi)}, \quad v = \frac{\Phi' + \Phi^2 - 2a\Phi}{2(a - \Phi)},$$

wherefrom we find the solution of (4.1.6)

$$y = \exp \int \frac{\Phi' - \Phi^2}{2(a - \Phi)}\,dx, \quad z = \exp \int \frac{\Phi' + \Phi^2 - 2a\Phi}{2(a - \Phi)}\,dx,$$

which contains an arbitrary differentiable function Φ.

REMARK. This method can be applied to any differential equation of the form

$$\frac{y''}{y} P\left(x, \frac{y'}{y}, \frac{z'}{z}\right) + \frac{z''}{z} Q\left(x, \frac{y'}{y}, \frac{z'}{z}\right) + R\left(x, \frac{y'}{y}, \frac{z'}{z}\right) = 0$$

where $\partial P(x_1, x_2, x_3)/\partial x_3 = \partial Q(x_1, x_2, x_3)/\partial x_2$. For details consult **[108]**.

4.5. *Comment.* Professor MITRINOVIĆ worked in undetermined differential equations by considering special equations arising in various applications. However, the methods he applied could be used for solving much wider classes of such equations. Though MITRINOVIĆ's results were generalised a number of times, it is surprising to note that no attempt was made to form some kind of a theory of those equations together with a classification of integrable types. The only exception is BANDIĆ's thesis **[109]**, in which the author made a systemathic research regarding some special classes of first and second order equations. A real generalisation of MITRINOVIĆ's methods has yet to be done.

Nevertheless, MITRINOVIĆ's results, though of a particular nature, proved to be important in applications, as was emphasized for instance by L'HERMITE **[110]**.

5. New integrable types of differential equations

5.1. *Reduction of certain nonlinear equations to a system of a linear and a nonlinear equation of lower order.* Papers **[44]**, **[45]**, **[46]** and **[47]** employ the same method of solution. In those papers MITRINOVIĆ reduced certain nonlinear equations to systems of a linear and a nonlinear equation of lower order. We shall illustrate the method by considering the equation

$$y^{(m)}+\sum_{k=1}^{m} a_k(x)\,y^{(m-k)}=F(x,\,z,\,z',\,\ldots,\,z^{(s)}), \tag{5.1.1}$$

where

$$z=y^{(n)}+\sum_{k=1}^{n} b_k(x)\,y^{(n-k)}, \tag{5.1.2}$$

which MITRINOVIĆ studied in **[46]**. Clearly, the equation (5.1.1) has order $H=\max(m,\,n+s)$. If we differentiate (5.1.2) $m-n$ times, and if we eliminate $y^{(m)},\ldots,y^{(n)}$ from (5.1.1), (5.1.2) and the obtained $m-n$ equations we arrive at an equation which connects $a_1,\ldots,a_m,\,b_1,\ldots,b_n,\,y,\ldots,y^{(n-1)},\,z,\ldots,z^{(m-n)}$ and F. If we now impose certain conditions on the coefficients $a_1,\ldots,a_m$, $b_1,\ldots,b_n$ such that the functions $a_1,\ldots,a_{m-n},\,b_1,\ldots,b_n$ remain arbitrary, the obtained equality takes the form of a differential equation

$$G(x,\,z,\,z',\,\ldots,\,z^{(h)})=0, \tag{5.1.3}$$

where $h=\max(m-n,\,s)$. Hence, the integration of (5.1.1) is reduced to the integration of the system consisting of (5.1.2) an (5.1.3).

Special equations of type (5.1.1) were considered in **[44]** and **[45]**. In **[44]** MITRINOVIĆ studied the equations

$$\Big(y^{(n)}+\sum_{k=1}^{n} a_k(x)\,y^{(n-k)}+a_{n+1}(x)\Big)\Big(y^{(n)}+\sum_{k=1}^{n} b_k(x)\,y^{(n-k)}+b_{n+1}(x)\Big)+c(x)=0$$

$$\Big(y^{(n)}+\sum_{k=1}^{n} a_k(x)\,y^{(n-k)}+a_{n+1}(x)\Big)\Big(y^{(n-1)}+\sum_{k=1}^{n-1} b_k(x)\,y^{(n-k-1)}+b_n(x)\Big)+c(x)=0$$

and proved that integration of those two equations can be reduced to the integration of a linear equation and an equation of ABEL's type, in the first case $u'=(\alpha_0u^2+\alpha_1u+\alpha_2)(u+\beta)^{-1}$ and in the second case $u'=\alpha_0+\alpha_1u+\alpha_2u^2+\alpha_3u^3$. The general result from **[46]** is also briefly mentioned in that paper.

In [45] MITRINOVIĆ considered the so-called ABEL's equations of higher order, i.e. equations of the form

$$y^{(n)}=\frac{P(x, y, y', \ldots, y^{(n-1)})}{Q(x, y, y', \ldots, y^{(n-1)})},$$

where P is a second order polynomial and Q a first order polynomial for $n=2$ and $n=3$. The obtained results are in connection with the results from [44].

Finally, in [47] MITRINOVIĆ considered the equation

$$(5.1.4) \qquad (y'+a_1(x)y+a_2(x))^p\,(y'+b_1(x)y+b_2(x))^q=f(x) \qquad (p, q\in \mathbf{Z})$$

whose integration is reduced to the integration of the system

$$(5.1.5) \qquad \frac{du}{dx}=\frac{\alpha_0 u^{p+q+1}+\alpha_1 u^{p+1}+\alpha_2 u}{\beta_0 u^{p+q}+\beta_1},$$

$$\frac{dy}{dx}+a_1(x)y+a_2(x)=u^q,$$

where $\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1$ depend on a_1, a_2, b_1, b_2. This means that the equations (5.1.4) and (5.1.5) are at the same time both integrable.

5.2. *Formation of integrable second order equations of prescribed type.* In papers [48], [49], [50] and [51] MITRINOVIĆ applied an elementary idea, but he obtained some quite general results. The idea is simply to replace the equation

$$(5.2.1) \qquad y''+A(x)y'+B(x)y=0$$

by the system

$$y'+f(x)y=z, \quad z'+g(x)z=0,$$

which is integrable for any pair of functions f and g. This replacement is possible on condition

$$f+g=A, \quad f'+fg=B.$$

If we investigate the integrability of (5.2.1) where the coefficients A and B are of given form, then the functions f and g should be adjusted so that A and B have the requested form. This leads to a criterion of integrability of (5.2.1).

MITRINOVIĆ applied this procedure to the equation

$$(5.2.2) \qquad y''+(ax+b)y'+(\alpha x^2+\beta x+\gamma)y=0$$

in which case we have $f(x)=px+q$, $g(x)=rx+s$, and to the equation

$$(5.2.3) \qquad y''+(ae^{kx}+b)y'+(\alpha e^{2kx}+\beta e^{kx}+\gamma)y=0$$

in which case $f(x)=pe^{kx}+q$, $g(x)=re^{kx}+s$, and also to the general equation (5.2.1). The results he obtained are generalisations of many known results, and he also arrived at new integrable types of equations (5.2.2) and (5.2.3).

5.3. *Miscellaneous equations.* In papers **[52]**—**[72]** special methods are applied to special equations with the aim to obtain their general solutions. We shall breifly mention the main result of each of those papers.

A generalisation of a method due to Appell is employed in **[52]** to form integrable equations of the form

$$y' = \sum_{k=0}^{m} a_k y^{(k)} \Big/ \sum_{k=1}^{n} b_k y^{(k)},$$

while in **[53]** it was investigated under what conditions can Liouville's equation

$$y' + f(x) \cos y + g(x) \sin y + h(x) = 0$$

be reduced to Lagrange's differential equation.

The following result was proved in **[54]**: If the functions $F(Z) = \psi(P, Q) + i\varphi(P, Q)$ and $f(z) = P(x, y) + iQ(x, y)$, where $Z = P + iQ$, $z = x + iy$, are analytic, then the equation

$$y' = \psi(P, Q)/\varphi(P, Q)$$

is integrable. A number of special cases is pointed out, e.g. the equations

$$y' = \operatorname{cotg} v(x, y), \quad y' = -\operatorname{tg} v(x, y), \text{ where } v_{xx} + v_{yy} = 0.$$

The general solution of the equation

$$y'' - 3\frac{f'(x)}{f(x)} y' + \left(3\frac{f'(x)^2}{f(x)^2} - \frac{f''(x)}{f(x)}\right) y = 0,$$

which appears in a problem of Mathematical Physics in found in **[55]** and in **[56]** an other equation which appears in practical problems, namely,

$$y'' + f(x) y' + g(x) y = 0,$$

with

$$f(x) = -B'(x)/B(x), \quad g(x) = a^{-2} B(x) \big(A_1(x) + A_2(x)\big) \big(A_1(x) A_2(x)\big)^{-1}$$

($a = \text{const}$) is studied.

The general solution of the equation

$$xy^{(n)} - mny^{(n-1)} + axy = 0 \quad (a \in \mathbf{R},\ m, n \in \mathbf{N})$$

was known, but Mitrinović **[57]** found that solution by a simple method, and expressed it in a simple form. Paper **[58]** contains sufficient conditions for the functions f and g so that the equation $f(x) y'' + g(x) y' + ax^n y = 0$ $(a, n = \text{const})$ is integrable.

In **[59]** it was shown that the equation

$$(y')^k + axy^{n-1} y' + by^n = 0 \qquad (a, b, k, n \in \mathbf{R})$$

is integrable in quadratures; in **[60]** it is established that the equations

$$y'' + (ax^2 + bx + c) y' + (Ax + B) y = 0$$

$$y'' + (ax^2 + bx + c) y' + \left((2a - A) x + b\left(1 - \frac{A}{a}\right) + B\right) y = 0$$

(a, b, c, A, $B=$const) which have the same form, are equivalent with respect to integrability, and in **[61]** some remarks regarding the EMDEN's equation

$$y''+\frac{2}{x}y'+y^n=0 \quad (n\in\mathbf{R})$$

are given.

Papers **[62]** — **[72]** carry the same title: *Compléments au traité de Kamke*, and they bring a number of general or special methods for obtaining new types of integrable differential equations.

Special equations were considered in papers **[62]**, **[65]**, **[66]**, **[67]**, **[68]** and **[69]**. So, for example, in **[62]** the equation

$$y''+\left(\frac{\alpha f(x)f'(x)}{f(x)^2+\beta}-\frac{f''(x)}{f'(x)}\right)y'+\frac{\gamma f'(x)^2}{f(x)^2+\beta}y=0$$

is reduced to the equation

$$\left(af(x)^2+1\right)z''+bf(x)z'+cz=0;$$

in **[65]** sufficient conditions for the integrability of the equation

$$x(A_1x^2+B_1xy+C_1y^2+D_1x+E_1y+F_1)\,\mathrm{d}y=$$
$$y(A_2x^2+B_2xy+C_2y^2+D_2x+E_2y+F_2)\,\mathrm{d}x$$

are given, while in **[68]** it is pointed out that $y=\sum_{k=0}^{m}a_kx^k\Big/\sum_{k=0}^{n}b_kx^k$ ($a_1,\ldots,a_m$, $b_1,\ldots,b_n$ arbitrary constants) is the general solution of the equation

$$\begin{vmatrix} y_{m-n+1} & y_{m-n+2} & \cdots & y_{m+1} \\ y_{m-n+2} & y_{m-n+3} & & y_{m+2} \\ \vdots & & & \\ y_{m+1} & y_{m+2} & & y_{m+n+1} \end{vmatrix}=0$$

where $y_k=(1/k!)\,y^{(k)}$.

A number of equations of the form

$$y^{(n+k)}-f(x)\left(\sum_{\nu=1}^{n}A_\nu x^{n-\nu}y^{(n-\nu)}\right)=0$$

whose solutions can be obtained by quadratures or special functions are presented in **[66]**. Paper **[67]** contains several particular results. The following equations are integrated:

$$x(Ax^4+Bxy+C)\,\mathrm{d}y+y(Ax^4+Exy+F)\,\mathrm{d}x=0,$$

$$(x^2y-1)y'-(xy^2-1)=0,$$

$$x(2y+x-1)y'-y(y+2x+1)=0,$$

$$(ae^{bx}+b^{-2})y''=y,$$

$$(xy'-ay)y''+by'^2=0,$$

and in **[69]** a linear fifth order equation is formed, with the property that its general solution is $y=\sum_{k=1}^{5} C_k u^{5-k} v^{k-1}$, where u and v are linearly independent solutions of the equation $z''=f(x)z$.

Other papers from this series are more general in character. In **[64]** the condition on $\alpha, \beta, \gamma, \lambda, \mu, \nu$ which ensures the integrability of the equation

$$x^\alpha y^\beta (y')^\gamma + A y^\lambda (y')^\mu + B x^\nu = 0$$

is determined. Mitrinović pointed out that this result contains 15 particular equations from Kamke's collection as special cases.

The possibility of determining integrating factors for the equations

$$x\left(ax^p y^q + f(xy)\right) \mathrm{d}y + y\left(bx^p y^q + g(xy)\right) \mathrm{d}x = 0,$$
$$x\left(ax^p y^q + f(x^r y^s)\right) \mathrm{d}y + y\left(bx^p y^q + g(x^r y^s)\right) \mathrm{d}x = 0,$$
$$x\left(a_1 x^p y^q + b_1 x^r y^s + f(xy)\right) \mathrm{d}y + y\left(a_2 x^p y^q + b_2 x^r y^s + g(xy)\right) \mathrm{d}x = 0,$$
$$x\left(a_1 x^p + b_1 x^2 y + c_1 xy^2 + d_1\right) \mathrm{d}y + y\left(a_2 x^p + b_2 x^2 y + c_2 xy^2 + d_2\right) \mathrm{d}x = 0$$

is studied in **[63]**, and this method is later developed, so that in **[70]** integrating factors for the equations

$$x\left(f(x^r y^s) + \sum_{k=1}^{n} a_k x^{p_k} y^{q_k}\right) \mathrm{d}y + y\left(g(x^r y^s) + \sum_{k=1}^{n} b_k x^{p_k} y^{q_k}\right) \mathrm{d}x = 0,$$
$$x\left(f(x^p y^q) + F(x^r y^s)\right) \mathrm{d}y + y\left(g(x^p y^q) + G(x^r y^s)\right) \mathrm{d}x = 0,$$
$$x\left(ax^p y^q + f(x^2 + cy^2)\right) \mathrm{d}y + y\left(bx^r y^s + g(x^2 + cy^2)\right) \mathrm{d}x = 0$$

are determined. A number of equations from Kamke's collection that can be integrated by this method is cited.

In paper **[71]** a linear fourth order equation is formed, such that its general solution is

$$y = C_1 u_1 v_1 + C_2 u_1 v_2 + C_3 u_2 v_1 + C_4 u_2 v_2,$$

where u_1, u_2 are linearly independent solutions of the equation $u''+f(x)u' + g(x)u = 0$, and v_1, v_2 are linearly independent solutions of the equation $v''+f_2(x)v' + g_2(x)v = 0$. Certain generalisations of this method are also given.

Finally, in **[72]** an interesting variant of the method of variation of parameters is applied to certain nonlinear differential equations. We illustrate the method by an example.

The differential equation

$$y'' + A(x)y' = F(x, y) \tag{5.3.1}$$

can be replaced by the system

$$y' + f(x)y = z, \quad z' + g(x)z = F(x, y), \tag{5.3.2}$$

where the functions f and g are determined by $f+g=A$, $f'+fg=0$.

Solving the "linear parts" of the system (5.3.2) we find

(5.3.3) $\quad y = Ce^{-\int f(x)\,dx}, \quad z = De^{-\int g(x)\,dx}$ (C, D arbitrary constants).

Suppose that C and D are differentiable functions of x, and substitute (5.3.3) into (5.3.2) to find

$$C' = De^{\int (f(x)-g(x))\,dx}, \quad D' = F\left(x,\ Ce^{-\int f(x)\,dx}\right) e^{\int g(x)\,dx},$$

and hence

(5.3.4)
$$\frac{dC}{dD} = \frac{De^{\int (f(x)-2g(x))\,dx}}{F\left(x,\ Ce^{-\int f(x)\,dx}\right)}.$$

Therefore, if the expression $F\left(x,\ Ce^{-\int f(x)\,dx}\right) e^{\int (2g(x)-f(x))\,dx}$ does not depend on x, but only on C, we can find a connection between C and D. This leads to the general solution of the equation (5.3.1).

Generalisations of this idea are also suggested in [**72**].

5.4. *Comment*. Papers of Professor MITRINOVIĆ devoted to the formation of new integrable types of differential equations seem to have made greater impact on other mathematicians than his other papers on differential equations. Almost all of his ideas served as starting points for numerous papers and Ph. D. theses, published both in Yugoslavia and abroad.

However, special tribute was paid to Professor MITRINOVIĆ by two distinguished mathematicians: E. KAMKE and F. G. TRICOMI.

In the preface to the sixth edition of his book [**104**], KAMKE drew special attention to MITRINOVIĆ's papers [**62**] — [**66**]. Following that recommendation, the Russian editions of KAMKE's book [**105**] contain an Appendix consisting of MITRINOVIĆ's papers [**62**] — [**67**], which are given in a shortened translation.

On the other hand, F. TRICOMI (see [**111**] and [**112**]) commenting on the current problems in the theory of differential equations, emphasized MITRINOVIĆ's approach to differential equations as one of the three contemporary trends of research, quoting his paper [**63**] as a representative of that approach.

We finally mention that there are many ideas in the papers of Professor MITRINOVIĆ which could be further developed or generalised in various directions.

6. Other papers

6.1. *Differential geometry*. In section 2.5 we have already mentioned certain papers of Professor MITRINOVIĆ in which he reduced various problems from differential geometry to the integration of the equation (2.1.1). In addition, MITRINOVIĆ published 14 more papers on differential geometry. Those are papers [**73**] — [**86**]. In all those papers he determined surfaces of the form $z = f(x, y)$ such that the projections of some special lines onto the xy-plane have a preassigned form.

So, for example, geodesic lines are considered in [**73**] and [**86**]. It is shown in [**73**] that surfaces $z = ky + f(x)$ or $z = kx + f(y)$ ($k = \text{const}$) have the property that their geodesic lines can be determined by quadratures. In [**86**] surfaces $z = u(x)\,y + v(x)$ are formed so that among the projections of their geodesic lines onto the xy-plane there is one given by $yu'(x) + v(x) = 0$.

Other papers are mainly concerned with asymptotic lines. For instance, in [**76**] surfaces with the equations

$$x=f_1(u, v), \quad y=f_2(u, v), \quad z=f_3(u, v)$$

are formed with the property that their differential equations for the asymptotic lines have simple form. It is shown that this is the case when f_1, f_2, f_3 satisfy the LAPLACE or the wave equation. In [**78**] a surface $z=F(x, y)$ is determined so that its differential equation for the asymptotic lines reads $f(y)(y')^2+2y'+g(x)=0$; in [**80**] surfaces $z+vy=f(x, v)$, $y=\partial f/\partial v$ such that they themselves together with their asymptotic lines are determined from the same equation; and in [**85**] functions f_k are found so that the differential equation of the asymptotic lines of the surface

$$z=\sum_{k=0}^{n} f_k(x)\, y^{n-k}$$

reduces to the equation (2.4.1).

6.2. *Partial differential equations*. Although he worked mainly in ordinary differential equations, Professor MITRINOVIĆ also published seven papers ([**87**]—[**93**]) on partial differential equations.

The equation

$$\left(\left(\frac{\partial f}{\partial \rho}\right)^2-1\right)\left(\frac{\partial \rho}{\partial \theta}\right)^2+2\frac{\partial f}{\partial \rho}\,\frac{\partial f}{\partial \theta}\,\frac{d\rho}{d\theta}+\left(\frac{\partial f}{\partial \theta}\right)^2-\rho^2=0$$

which has a geometrical interpretation, is integrated in [**87**] in some special cases.

Papers [**88**], [**89**], [**90**] treat similar problems. In [**89**] the following equation is considered:

$$G(F_1, F_2, \ldots, F_{n+1})=0, \tag{6.2.1}$$

where

$$F_k=F_k\left(x_1, \ldots, x_n, z, \frac{\partial z}{\partial x_1}, \ldots, \frac{\partial z}{\partial x_n}\right),$$

and the following result it proved:

If the system $F_k=C_k$ ($k=1, \ldots, n+1$; C_k arbitrary constants) is in involution, then the complete integral of (6.2.1) is given by

$$\begin{cases} F_k=C_k \quad (k=1, \ldots, n+1) \\ G(C_1, \ldots, C_{n+1})=0. \end{cases}$$

Paper [**88**] is a summary of [**90**]. Two linear equations which contain the expression

$$(xp+yq)^{(k)}=\sum_{\nu=0}^{k}\binom{k}{\nu} x^{k-\nu} y^{\nu}\frac{\partial^k z}{\partial x^{k-\nu}\partial y^{\nu}},$$

are considered in [**90**]. They are:

$$\sum_{\nu=1}^{n}\alpha_\nu (xp+yq)^{(n-\nu)}=0 \tag{6.2.2}$$

and its special case:

$$z = \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k!} (xp + yq)^{(k)}. \tag{6.2.3}$$

The general solution of (6.2.2) containing n arbitrary functions of y/x is found, and it is also proved that the complete integral of (6.2.3) can be obtained by replacing the derivatives by arbitrary constants.

D'ALEMBERT's method for obtaining particular solutions of ordinary linear differential equations with constant coefficients in the case when the characteristic equation has multiple roots in applied in **[92]** to the equation

$$au_{xx} + 2\,bu_{xy} + cu_{yy} = 0,$$

in the case when the characteristic equation $a + 2bt + ct^2 = 0$ has a double root. The method is then extended to linear partial differential equations of higher order.

Finally, papers **[91]** and **[93]** are concerned with undetermined equations. The method exposed in section 4.3 is applied to certain partial differential equations (see **[91]**) such as

$$vu_{xy} = kuv_{xy}, \quad v_y u_{xxx} = ku_y v_{xxx} \quad (k = \text{const}).$$

Some generalisations are suggested in **[93]**.

6.3. *Miscellaneous results.* In this final section we mention six more papers of Professor MITRINOVIĆ which could not be classified in above sections.

In **[94]** a result due to LAGRANGE and SERRET, regarding first integrals of certain differential equations is given in an other form. First integrals are again considered in **[95]** where a result of M. PETROVIĆ is generalised.

A VOLTERRA type integral equation

$$g(x) = f(x) - \int_0^x (a(x) - a(y)) f(y)\,dy \tag{6.3.1}$$

can be, under certain assumptions, be reduced to the differential equation

$$g(x) = \left(\frac{z'}{a'}\right)' - z \qquad (z = f - g).$$

In **[96]** MITRINOVIĆ showed that the equation

$$\left(\frac{z'}{f(x)}\right)' - z = 0$$

is integrable if and only if the equation $z'' - F(x)\,z = 0$ is integrable. Hence, it is established that there exists an infinite sequence of equations (6.3.1) whose solutions can be obtained in closed form.

Papers **[97]**, **[98]** and **[99]** are bibliographical notes, discussing questions of priority for the equation $au_{xx} - u_t = 0$ (**[97]**), the equation $\sum_{k=0}^{n} \binom{m}{k} \frac{d^k p_m(x)}{dx^k} \frac{d^{n-k}}{dx^{n-k}} y =$ $= 0$ (**[98]**) and for a result in differential geometry (**[99]**).

7. Conclusion

In previous sections we have briefly mentioned and commented on the research work in differential equations of Professor MITRINOVIĆ. A detailed analysis of his contributions would require much more space. However, something still remains to be said about his pedagogical work.

Professor MITRINOVIĆ published two text-books ([**100**], [**101**]) on differential equations. Besides that, many of his collections of problems contain substanial number of problems on differential equations. They were finally collected and published in a separate book [**102**].

It is also worth noticing that 6 mathematicians took their Ph. D. theses in differential equations with Professor MITRINOVIĆ. They are: B. S. POPOV (1952), I. BANDIĆ (1958), D. PERČINKOVA (1963), I. ŠAPKAREV (1964), J. D. KEČKIĆ (1970), P. R. LAZOV (1977).

It is surprising, and it is a pity, that Professor MITRINOVIĆ never wrote a monograph on closed form solutions of ordinary differential equations. The reason is that during the last fifteen years (which were his main „book-writing period") Professor MITRINOVIĆ almost completely turned his attention to other disciplines (mainly inequalities). Nevertheless, there is time.

We conclude this paper by a remark of a private nature. More than once Professor MITRINOVIĆ said to the present author that he made a mistake in spending so much time on such an ungratifying discipline as the integration of differential equations is well known to be, and that he would have done better if he had devoted all his time to some other field (e.g. inequalities). The author of this paper wishes to express a profound disagreement with that opinion of Professor D. S. MITRINOVIĆ.

REFERENCES

D. S. MITRINOVIĆ:

1.* *Nouveaux cas d'intégrabilité d'une équation différentielle du premier ordre.* Bull. Acad. Sci. Math. Nat. Belgrade **1** (1933), 107—117.

2.* *Nouvelles formes intégrable d'une équation différentielle importante du premier ordre.* Ibid. **2** (1935), 61—65.

3.** *Истраживања о једној важној диференцијалној једначини првог реда.* Београд 1935, 39 стр.

4.* *Sur l'intégration d'une équation différentielle importante du premier ordre.* Bull. Acad. Sci. Math. Nat. Belgrade **3** (1936), 7—19.

5. *Intégration d'une équation différentielle du premier ordre et polinômes d'Hermite qui s'y rattachent.* Revista de Ciencias, Lima **38** (1936), 123—127.

6. *O jednoj diferencijalnoj jednačini* (with I. VIDAV). Bull. Soc. Math. Phys. Macédoine **1** (1950), 21—27.

7. *Sull'integrazione dell'equatione differenziale del tipo di Abel.* Rendiconti del Reale Istituto Lombardo di scienze e lettere (2) **69** (1936), 203—208.

8.* *Sur une classe d'équations différentielles du premier ordre que l'on rencontre dans divers problème de Géométrie.* Bull. Acad. Sci. Math. Nat. Belgrade **6** (1939), 99—120.

9. *Sur une équation différentielle du premier ordre intervenant dans divers problèmes de géométrie.* C. R. Acad. Sci. Paris **204** (1937), 1706—1708.

* Papers marked by * also appeared in Serbian in the journal Глас Српске Краљевске Академије.

** This is the thesis of Professor D. S. MITRINOVIĆ.

10. *Sur l'équation différentielle des lignes géodésiques des surfaces spirales.* Ibid. **205** (1937), 1194—1196.

11. *Sur une équation différentielle du premier ordre intervenant dans divers problèmes de géométrie.* Bulletin des Sciences mathématiques (2) **61** (1937), 323—325.

12. *Transformation et intégration d'une équation différentielle du premier ordre.* Publications mathématiques de l'Université de Belgrade **5** (1936), 10—22.

13. *Sur l'équation différentielle des lignes asymptotiques.* Ibid. **3** (1934), 175—178.

14. *Sur certaines surfaces dont les lignes asymptotiques se déterminent par quadratures* (with R. Godeau). Mathesis **51** (1937), 115—116.

15. *Asymptotiques d'une classe de surfaces et équations différentielles linéaires du second ordre s'y rattachant.* Bulletin de l'Académie royale de Belgique (5) **22** (1936), 948—950.

16.* *Recherches sur les lignes asymptotiques.* Bull. Acad. Sci. Math. Nat. Belgrade **4** (1938), 105—120.

17. *Sur les lignes de courbure des surfaces réglées à plan directeur.* Publications mathématiques de l'Université de Belgrade **5** (1936), 100—102.

18. *Un problème sur les lignes asymptotiques et la méthode de l'intégration logique des équations différentielles de Jules Drach.* C. R. Acad. Sci. Paris **205** (1937), 1358—1360.

19. *Problèmes géométriques où interviennent diverses équations différentielles.* Ibid. **206** (1938), 568—570.

20. *Sur le problème de Beltrami: déformer une surface réglée de telle manière que l'une de ses courbes, assignée a l'avance devienne planne.* Bulletin des Sciences mathématiques (2) **63** (1939), 99—105.

21. *Remarque sur une équation différentielle du premier ordre.* Publications mathématiques de l'Université de Belgrade **3** (1934), 167—170.

22. *Théorèmes relatifs à l'équation différentielle de Riccati.* C. R. Acad. Sci. Paris **206** (1938), 411—413.

23.* *Quelques propositions relatives à l'équation différentielle de Riccati.* Bull. Acad. Sci. Math. Nat. Belgrade **6** (1939), 121—156.

24. *Compléments au Traité de Kamke XII. Des critères d'intégrabilité de l'équation différentielle de Riccati* (with P. M. Vasić). Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. № **175**—№ **179** (1967), 15—21.

25. *Dopune Kamkeovom delu XIII. O kriterijumima integrabilnosti Riccatieve jednačine* (with P. M. Vasić). Ibid. № **210**—№ **228** (1968), 43—48.

26. *Sur un problème de Darboux.* Bulletin de la Section scientifique de l'Académie Roumaine **20** (1938), 23—25.

27. *Théorème sur l'équation de Riccati.* C. R. Acad. Sci. Paris **208** (1939), 156—157.

28. *A propos d'une Note de D. Pompeiu relative à l'équation de Riccati.* Bulletin de la Section scientifique de l'Académie Roumaine **30** (1947), 256—263.

29. *Sur une équation différentielle linéaire du second ordre transformable en elle-même.* C. R. Acad. Sci. Paris **228** (1949), 1188—1190.

30. *О једној класи Riccati-евих једначина које су инваријантне у односу на једну смену функције.* Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **2** (1949), 135—139.

31. *Mise en correspondance d'un problème non résolu de la théorie de l'élasticité avec un problème résolu par Darboux et Drach.* C. R. Acad. Sci. Paris **231** (1950), 327—328.

32. *O diferencijalnoj jednačini jednog važnog problema teorije elasticiteta.* Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **3** (1950), No. 1, 19 pp.

33. *O jednoj neodređenoj diferencijalnoj jednačini.* Ibid. **3** (1950), No. 6, 16 pp.

34. *Sur un procédé d'intégration d'une équation de Monge.* C. R. Acad. Sci. Paris **232** (1951), 1334—1336.

35. *Treća metoda integracije Nemenyi-Truesdellove jednačine.* Bull. Soc. Math. Phys. Macédoine **2** (1951), 17—20.

36. *On an equation of Nemenyi and Truesdell.* J. Washington Acad. Sci. **41** (1951), 123.

37. *Sur une équation différentielle indéterminée du second ordre.* Bulletin de l'Académie royale de Belgique (5) **37** (1951), 227—228.

38. *Sur l'équation différentielle d'un problème d'Hydrodynamique.* C. R. Acad. Sci. Paris **241** (1955), 1708—1710.

39. *Sur l'équation différentielle d'un problème de technique étudié par R. Gran Olsson.* Norske Vid. Selsk. Forh. (Trondheim) **28** (1955), 171—175.

40. *O nekim neodređenim diferencijalnim jednačinama.* Bull. Soc. Math. Phys. Serbie **7** (1955), 171—178.

41. *Sur l'équation différentielle de Sommerfeld pour la stabilité hydrodynamique.* C. R. Acad. Sci. Paris **242** (1956), 2287—2289.

42. *Sur une équation différentielle indéterminée intervenant dans un problème important de l'Elasticité.* Ibid. **232** (1951), 681—683.

43. *Sur un opérateur différentiel.* La Revue scientifique, Paris **89** (1951), 44.

44. *Sur une classe d'équations différentielles.* Bulletin des Sciences mathematiques (2) **62** (1938), 36—41.

45.* *Équations différentielles d'Abel d'ordre supérieur.* Bull. Acad. Sci. Math. Nat. Belgrade **5** (1939), 25—31.

46. *Sur une classe d'équations différentielles d'ordre supérieur.* Bulletin de l'Académie royale de Belgioue (5) **33** (1947), 521—526.

47. *О шрансформацији једне диференцијалне једначине.* Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **1** (1948), 97—109.

48. *Посшуйак за формирање криширијума иншеīрабилишеша линеарних диференцијалних једначина чији коефицијенши имају облике унаџред даше.* Ibid. **2** (1949), 165—182.

49. *Sur un cas de réductibilité d'équations différentielles linéaires.* C. R. Acad. Sci. Paris **230** (1950), 1130—1132.

50. *Sur un procédé fournissant des équations différentielles linéaires intégrables d'un type assigné d'avance.* Publ. Inst. Math. (Beograd) **3** (1950), 227—234.

51. *Povodom Görtlerovih rezultata o linearnoj diferencijalnoj jednačini drugog reda.* Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **3** (1950), No. 1, 19 pp.

52.* *Cas d'intégrabilité d'une certaine classe d'équations différentielles algébriques du premier ordre.* Bull. Acad. Sci. Math. Nat. Belgrade **2** (1935), 247—248.

53. *Contributions à l'intégration de l'équation différentielles de J. Liouville.* Publications mathématiques de l'Université de Belgrade **4** (1935), 149—152.

54. *Sur l'emploi de la partie réelle et de la partie imaginaire des fonctions analytiques dans l'intégration des équations différentielles.* The Tôhoku Mathematical Journal **42** (1936), 179—184.

55. *О једној диференцијалној једначини друīоī реда која се йојављује у једном йроблему машемашичке физике.* Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **2** (1949), 187—193.

56. *Sur l'équation différentielle d'un probleme de Kuhelj.* Bull. Soc. Math. Phys. Macédoine **2** (1951), 31—34.

57. *Sur équation différentielle de Laplace.* Ibid. **2** (1951), 109—112.

58. *Sur l'équation différentielle d'Emden generalisée.* C. R. Acad. Sci. Paris **241** (1955), 724—726.

59. *Sur une équation différentielle du premier ordre.* Jber. Deutsch. Math.-Verein. Abt. **2**, **58** (1955), 1.

60.*** *Problèmes sur une équation différentielle* . Mathesis **69** (1960), 223—224.

61. *Sur l'équation d'Emden* (with K. Slipičević) Ibid. **69** (1960), 74—75.

62. *Compléments au Traité de Kamke, II.* Bull. Soc. Math. Phys. Serbie **7** (1955), 161—164.

63. *Compléments au Traité de Kamke, I.* Jber. Deutsch. Math.-Verein. Abt. 2, **58** (1956), 58—60.

64. *Compléments au Traité de Kamke, III.* Bol. Un. Mat. Ital. (3) **11** (1956), 168—171.

65. *Compléments au Traité de Kamke, IV.* Glasnik matematičkofizički i astronomski **11** (1956), 7—10.

*** Also appeared in Serbian in Bull. Soc. Math. Phys. Serbie.

66. *Compléments au Traité de Kamke, V.* Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. № **11** (1957), 10 pp.
67. *Compléments au Traité de Kamke, VI.* Ibid. № **27** (1959), 4 pp.
68. *Compléments au Traité de Kamke, VIII.* Ibid. № **84**—№ **91** (1963), 19—20.
69. *Compléments au Traité de Kamke, IX* (with D. Ž. Đoković). Ibid. № **107**—№ **108** (1963), 78—79.
70. *Compléments au Traité de Kamke, X* (with P. M. Vasić). Publ. Inst. Math. (Beograd) **3** (17) (1963), 61—68.
71. *Compléments au Traité de Kamke, XI* (with P. M. Vasić). Mat. Vesnik **1** (16) (1964), 181—185.
72. *Compléments au Traité de Kamke, XIV. Applications of the variations of parameters method to nonlinear second order differential equations* (with J. D. Kečkić). Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. № **544**—№ **576** (1976), 3—7.
73. *Sur les lignes géodésiques d'une classe des surfaces.* Publications mathématiques de l'Université de Belgrade **3** (1934), 167—170.
74. *Sur certaines trajectoires des courbes algébriques planes de genre zéro, un et deux.* Ibid. **4** (1935), 153—160.
75. *Remarques sur les lignes asymptotiques et sur lignes de courbure.* Praktika de l'Académie d'Athenes **10** (1935), 480—483.
76. *Équation différentielle des asymptotiques et équation des cordes vibrantes qui s'y rattache.* Rev. math. Union Interbalkan. **1** (1936), 53—57.
77. *Remarques sur les surfaces de translation.* Praktika de l'Académie d'Athenes **11** (1936), 356—359.
78. *Asymptotiques d'une classe des surfaces.* Bulletin de l'Académie royale de Belgique (5) **22** (1936), 948—950.
79. *Théorème sur les lignes asymptotiques.* Mathesis **50** (1936), 367—368.
80. *Un problème sur les lignes asymptotiques d'une classe de surfaces.* Bulletin de l'Académie royale de Belgique (5) **23** (1937), 323—325.
81. *Sur l'étude des lignes de courbure en coordonnées tangentielles.* C. R. Seances Conferences Soc. Math. France, 26 mai 1937, p. 32.
82. *Sur l'équation différentielles des lignes de courbure.* Publications mathématiques de l'Université de Belgrade **6—7** (1938), 32—35.
83.* *Problème, dont la solution dépend d'une équation de Riccati relatif aux asymptotiques d'une surface réglée.* Bull. Acad. Sci. Math. Nat. Belgrade **5** (1939), 89—92.
84. *Sur les lignes asymptotiques.* Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. № **115**—№ **121** (1963), 1—4.
85. *Lignes asymptotiques d'une classe des surface.* Ibid. No. **247**—No.**273** (1969), 53—56.
86. *On geodesic lines of a class of surfaces, II.* Ibid. № **302**—№ **319** (1970), 29—31.
87.* *Sur l'équation différentielle des courbes planes dont l'arc est une fonction donnée des coordonnées polaires.* Bull. Acad. Sci. Math. Nat. Belgrade **2** (1935), 245.
88. *Sur l'intégration d'une équation linéaire aux dérivées partielles.* C. R. Acad. Sci. Paris **210** (1940), 783—785.
89. *Remarque sur certaines équations aux dérivées partielles.* Atti del Reale Istituto Veneto di scienze, lettere ed arti **99** (1940), parte II, 357—360.
90. *O jeдној линеарној парцијалној jeдначини.* Glasnik matematičko-fizički i astronomski **1** (1946), 168—181 and 209—226.
91. *Sur certaines équations aux dérivées partielles à deux fonctions inconnues.* Bull. Soc. Math. Phys. Serbie **8** (1956), 3—6.
92. *Sur une équation linéaire aux dérivées partielles à coefficients constants.* The Mathematical Gazette **41** (1957), 41—43.
93. *Sur quelques équations aux dérivées partielles à deux fonctions inconnues.* Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. № **247**—№ **273** (1969), 57—60.
94.* *Nouvelle forme de la remarque de Lagrange-Serret relative aux équations différentielles ordinaires.* Bull. Acad. Sci. Math. Nat. Belgrade **3** (1936), 37—39.

95.* *Contribution à la théorie des intégrales premières d'équations différentielles.* Ibid. **3** (1936), 33—35.

96.* *Correspondance entre l'équation différentielle du second ordre et une équation intégrale de Volterra.* Ibid. **7** (1941), 191—195.

97. *Sur la solution de Ribaud de l'équation de Fourier.* Bull. Soc. Math. Phys. Macédoine **2** (1951), 105—107.

98. *Dopune Kamkeovom delu, VII.* Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. № **78**—№ **83** (1962), 16—18.

99. *Sur une note de Coşniţă relative aux trajectoires isogonales des famillies de cercles.* Ibid. № **84**—№ **91** (1963), 8.

100. *Jednačine matematičke fizike* (with J. D. Kečkić). Beograd 1972, XII+228 pp. Second edition: Beograd 1978, VIII+240 pp.

101. *Predavanja o diferencijalnim jednačinama.* Beograd 1976, XVI+226 pp.

102. *Diferencijalne jednačine — Zbornik zadataka i problema* (in cooperation with P. M. Vasić). Beograd 1972, XII+254 pp. Second edition: Beograd 1979, XII+266 pp.

E. Kamke:

103. *Differentialgleichungen. Lösungsmethoden und Lösungen.* Leipzig 1942.

104. *Differentialgleichungen. Lösungsmethoden und Lösungen,* 6. Auffl. Leipzig 1959.

Э. Камке:

105. *Справочник по обыкновенным дифференциальным уравнениям.* Москва 1971.

A. Buhl:

106. *Nouveaux éléments d'analyse,* tome III. Paris 1940.

В. А. Добровольский:

107. *Очерки развития аналитической теории дифференциальных уравнении.* Киев 1974.

J. D. Kečkić:

108. *On a class of undetermined differential equations.* Math. Balkanica **8** (1978).

I. Bandić:

109. *Metode rešavanja neodređenih diferencijalnih jednačina koje se javljaju u teoriji elasticiteta, hidrodinamici i elektronici.* Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. № **24** (1959), 40 pp.

R. L'Hermite:

110. *Résistance des matériaux,* tome I. Paris 1954.

F. G. Tricomi:

111. *Les problèmes actuels dans la théorie des équations différentielles ordinaires.* Bull. Soc. Math. Phys. Serbie **9** (1957), 169—178.

112. *Aktuelle Fragen in der Theorie der gewöhnlichen Differentialgleichungen.* MTW Mitteilungen Jahrg. 1958, No. **5—6**, 173—185.