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582. ON THE RATIO OF MEANS*

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In this article a result due to A. W. MARSHALL, I. OLKIN and F. PROSCHAN is generalized by providing a proof for the corresponding result for weighted means. The method different from that used by the aforementioned authors also enables further improvement of their inequality at the expense of some additional assumptions on the sequences. Integral analogous of these results have also been included.

Theorem 1. *If* $p_1>0, \ldots, p_n>0$, $a_1>0, \ldots, a_n>0$, $b_1 \geqq \cdots \geqq b_n>0$ *and if* $\frac{b_1}{a_1} \leqq \cdots \leqq \frac{b_n}{a_n}$, *then function* f, *defined by*

$$f(r)=\left(\frac{\sum_{i=1}^{n} p_i a_i^r}{\sum_{i=1}^{n} p_i b_i^r}\right)^{\frac{1}{r}} \qquad (r\neq 0,\ |r|<+\infty),$$

$$=\left(\frac{\prod_{i=1}^{n} a_i^{p_i}}{\prod_{i=1}^{n} b_i^{p_i}}\right)^{1\Big/\sum_{i=1}^{n} p_i} \qquad (r=0),$$

is nondecreasing.

Proof. We shall use the following inequality for means (see [1], p. 76):

$$\left(\frac{\sum_{i=1}^{n} P_i x_i^r}{\sum_{i=1}^{n} P_i}\right)^{\frac{1}{r}} \geqq \left(\frac{\sum_{i=1}^{n} P_i x_i^s}{\sum_{i=1}^{n} P_i}\right)^{\frac{1}{s}} \qquad (r\geqq s), \tag{1}$$

where $P_i,\ x_i>0 \quad (i=1, \ldots, n)$.

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Substituting: $P_i = p_i b_i^s$, $x_i = \frac{a_i}{b_i}$ $(i=1, \ldots, n)$ in (1) we obtain

$$\left(\frac{\sum_{i=1}^{n} p_i b_i^{s-r} a_i^r}{\sum_{i=1}^{n} p_i b_i^s} \right)^{\frac{1}{r}} \geqq \left(\frac{\sum_{i=1}^{n} p_i a_i^s}{\sum_{i=1}^{n} p_i b_i^s} \right)^{\frac{1}{s}} \quad (r \geqq s). \tag{2}$$

To complete the proof we must prove

$$\left(\frac{\sum_{i=1}^{n} p_i b_i^{s-r} a_i^r}{\sum_{i=1}^{n} p_i b_i^s} \right)^{\frac{1}{r}} \leqq \left(\frac{\sum_{i=1}^{n} p_i a_i^r}{\sum_{i=1}^{n} p_i b_i^r} \right)^{\frac{1}{r}} \quad (r \geqq s). \tag{3}$$

Let us take ČEBYŠEV's inequality

$$\sum_{i=1}^{n} q_i \sum_{i=1}^{n} q_i x_i y_i \geqq \sum_{i=1}^{n} q_i x_i \sum_{i=1}^{n} q_i y_i \tag{4}$$

which is true if $(q_1, \ldots, q_n)$ is a positive sequence and if both sequences $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are nonincreasing, or both nondecreasing. If one of the sequences $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ is nonincreasing and the other nondecreasing, the reverse inequality in (4) is valid (see [**1**], p. 36).

Substituting $q_i = p_i a_i^r$, $x_i = \left(\frac{b_i}{a_i}\right)^r$, $y_i = b_i^{s-r}$ in (4), we get for $r > 0$:

$$\sum_{i=1}^{n} p_i b_i^s \sum_{i=1}^{n} p_i a_i^r \geqq \sum_{i=1}^{n} p_i b_i^r \sum_{i=1}^{n} p_i b_i^{s-r} a_i^r \tag{5}$$

and for $r < 0$ we get the reverse inequality (as $(b_i/a_i)_{i=1,\ldots,n}$ is a nondecreasing sequence $(b_i^r/a_i^r)_{i=1,\ldots,n}$ is nondecreasing for $r > 0$ and nonincreasing for $r < 0$; as $(b_i)_{i=1,\ldots,n}$ is a nonincreasing sequence, $(b_i^{s-r})_{i=1,\ldots,n}$ is nondecreasing).

From (5) we obtain (3) for $r > 0$. If $r < 0$, from the reverse inequality related to (5) it follows (3).

The proof is, in fact, given for $rs \neq 0$. For $rs = 0$ the theorem can be proved directly by the transition to the limit.

Similarly, we can prove

Theorem 2. *If* $p_1 > 0, \ldots, p_n > 0$, $a_1 > 0, \ldots, a_n > 0$, $b_n \geqq b_{n-1} \geqq \cdots \geqq b_1 > 0$, *and if* $\frac{b_1}{a_1} \geqq \cdots \geqq \frac{b_n}{a_n}$, *then function* f *defined in Theorem* 1 *is nondecreasing.*

Substituting $p_i = 1$ $(i = 1, \ldots, n)$ in theorem 1 we get the result from [**3**]. The same result is there proved in a different way. The method we had used enables some generalizations.

Theorem 3. *If $p_1>0, \dots, p_n>0$, $x_1=0$, $x_2>0, \dots, x_n>0$ and $(0, x_2, \dots, x_n)$ is a convex sequence, then for $r \geqq s$, the following inequality is valid*

$$\left(\frac{\sum_{i=1}^{n} p_i x_i^r}{\sum_{i=1}^{n} p_i}\right)^{\frac{1}{r}} \geqq \alpha \left(\frac{\sum_{=1}^{n} p_i x_i^s}{\sum_{i=1}^{n} p_i}\right)^{\frac{1}{s}}, \tag{6}$$

where

$$\alpha = \left(\frac{\sum_{i=1}^{n} p_i}{\sum_{i=1}^{n} p_i (i-1)^s}\right)^{\frac{1}{s}} \left(\frac{\sum_{i=1}^{n} p_i (i-1)^r}{\sum_{i=1}^{n} p_i}\right)^{\frac{1}{r}} \geqq 1.$$

Proof. If we put $x_i = i-1$ in inequality (1), we obtain $\alpha \geqq 1$. To prove (6), we carry out substitutions in (1): $P_i = p_i (i-1)^s$, $x_i \to \frac{x_i}{i-1}$ $(i=1, \dots, n)$, and we get

$$\left(\frac{\sum_{i=1}^{n} p_i x_i^r (i-1)^{s-r}}{\sum_{i=1}^{n} p_i (i-1)^s}\right)^{\frac{1}{r}} \geqq \left(\frac{\sum_{i=1}^{n} p_i x_i^s}{\sum_{i=1}^{n} p_i (i-1)^s}\right)^{\frac{1}{s}}.$$

To complete the proof, we must show that

$$\left(\frac{\sum_{i=1}^{n} p_i x_i^r}{\sum_{i=1}^{n} p_i (i-1)^r}\right)^{\frac{1}{r}} \geqq \left(\frac{\sum_{i=1}^{n} p_i x_i^r (i-1)^{s-r}}{\sum_{i=1}^{n} p_i (i-1)^s}\right)^{\frac{1}{r}},$$

i. e.

$$\sum_{i=1}^{n} p_i x_i^r \sum_{i=1}^{n} p_i (i-1)^s \geqq \sum_{i=1}^{n} p_i (i-1)^r \sum_{i=1}^{n} p_i x_i^r (i-1)^{s-r},$$

which we get when we carry out substitutions in inequality (4):

$$q_i = p_i (i-1)^s,\ x_i = (i-1)^{r-s},\ y_i = \left(\frac{x_i}{i-1}\right)^r \quad (i=2, \dots, n),\ x_1 = y_1 = 0$$

$\Big($as $(x_i)_{i=1,\dots,n}$ is a convex sequence, $\left(\frac{x_i}{i-1}\right)_{i=2,\dots,n}$ is a nondecreasing sequence (see [**2**])$\Big)$.

Since $\alpha \geqq 1$, it may be concluded that inequality (6) is sharper than inequality (1) provided that the conditions of convexity for the sequence $(x_1, \dots, x_n)$ are added.

The following theorem is a generalization of theorem 1.

Theorem 4. *If* $p_1>0, \dots, p_n>0$, $a_1=0$, $a_2>0$, $a_3>0, \dots, a_n>0$, $b_1>0, \dots, b_n>0$, $b_1\leqq b_2\leqq \cdots \leqq b_n$ *and* $\left(\frac{a_1}{b_1}, \dots, \frac{a_n}{b_n}\right)$ *is a convex sequence, then for* $r\geqq s$

$$\left(\frac{\sum_{i=1}^{n} p_i a_i^r}{\sum_{i=1}^{n} p_i b_i^r}\right)^{\frac{1}{r}} \geqq \beta \left(\frac{\sum_{i=1}^{n} p_i a_i^s}{\sum_{i=1}^{n} p_i b_i^s}\right)^{\frac{1}{s}} \tag{7}$$

is valid, where

$$\beta = \left(\frac{\sum_{i=1}^{n} p_i b_i^r (i-1)^r}{\sum_{i=1}^{n} p_i b_i^r}\right)^{\frac{1}{r}} \left(\frac{\sum_{i=1}^{n} p_i b_i^s}{\sum_{i=1}^{n} p_i b_i^s (i-1)^s}\right)^{\frac{1}{s}} \geqq 1.$$

Proof. Putting $p_i \to p_i b_i^s$, $x_i \to \frac{a_i}{b_i}$ in (6) we obtain

$$\left(\frac{\sum_{i=1}^{n} p_i b_i^{s-r} a_i^r}{\sum_{i=1}^{n} p_i b_i^s (i-1)^r}\right)^{\frac{1}{r}} \geqq \left(\frac{\sum_{i=1}^{n} p_i a_i^s}{\sum_{i=1}^{n} p_i b_i^s (i-1)^s}\right)^{\frac{1}{s}} \qquad (r\geqq s).$$

To complete the proof, we must prove that

$$\sum_{i=1}^{n} p_i b_i^s (i-1)^r \sum_{i=1}^{n} p_i a_i^r \geqq \sum_{i=1}^{n} p_i b_i^r (i-1)^r \sum_{i=1}^{n} p_i b_i^{s-r} a_i^r,$$

which we get when we carry out substitutions in (4):

$$q_i = p_i b_i^s (i-1)^r, \quad x_i = b_i^{r-s}, \quad y_i = \left(\frac{a_i}{b_i (i-1)}\right)^r.$$

Since from (1) follows that $\beta\geqq 1$ inequality (7) is sharper than inequality (1).
An integral analogue to theorem 1 is given by

Theorem 5. *Let* p, f, g *be positive and integrable functions on* $[a, b]$. *If* $x \mapsto g(x)$ *and* $x \mapsto \frac{g(x)}{f(x)}$ *are monotone in the opposite sense, function* F, *defined by*

$$F(r) = \left(\frac{\int_a^b p(x) f(x)^r \, \mathrm{d}x}{\int_a^b p(x) g(x)^r \, \mathrm{d}x}\right)^{\frac{1}{r}} \qquad (r\neq 0; \quad |r| < +\infty),$$

$$F(0)=\exp\left(\frac{\int_a^b p(x)\log f(x)\,\mathrm{d}\,x}{\int_a^b p(x)\log g(x)\,\mathrm{d}\,x}\right),$$

is nonincreasing.

The preceding theorem as well as the next one can be proved analogously to the corresponding theorems in the discrete case.

Theorem 6. *If p, f, g are nonnegative functions, $x\mapsto \frac{f(x)}{g(x)}$ is a convex function on $[a, b]$, $f(a)=0$ and g is an increasing positive function on $[a, b]$, then*

$$\left(\frac{\int_a^b p(x)f(x)^r\,\mathrm{d}\,x}{\int_a^b p(x)\,g(x)^r\,\mathrm{d}\,x}\right)^{\frac{1}{r}}\geqq M\left(\frac{\int_a^b p(x)f(x)^s\,\mathrm{d}\,x}{\int_a^b p(x)\,g(x)^s\,\mathrm{d}\,x}\right)^{\frac{1}{s}},$$

where

$$M=\left(\frac{\int_a^b p(x)\,g(x)^r(x-a)^r\,\mathrm{d}\,x}{\int_a^b p(x)\,g(x)^r\,\mathrm{d}\,x}\right)^{\frac{1}{r}}\left(\frac{\int_a^b p(x)\,g(x)^s\,\mathrm{d}\,x}{\int_a^b p(x)\,g(x)^s(x-a)^s\,\mathrm{d}\,x}\right)^{\frac{1}{s}}\geqq 1.$$

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