UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. Fiz. № 577—№ 598 (1977), 30—32.

581. ADDITIONS TO KAMKE'S TREATISE, VIII: ON SINGULAR SOLUTIONS OF GENERALISED CLAIRAUT'S EQUATION*

Jovan D. Kečkić

1. CLAIRAUT's differential equation

$$y = x\,y' + f(y') \tag{1}$$

was generalised a number of times (see, for example, [1], [2], [3], eq. 6.248). The obtained result reads:

The n-th order CLAIRAUT's differential equation

$$y = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!}\, x^k\, y^{(k)} + f(y^{(n)}) \tag{2}$$

has the general solution

$$y = \sum_{k=1}^{n} \frac{1}{k!}\, C_k\, x^k + f(C_n), \tag{3}$$

where $C_1, \ldots, C_n$ are arbitrary constants.

For $n=1$ this result reduces to the classical result regrading CLAIRAUT's equation (1).

However, equation (1) also possesses the singular solution, obtained by the elimination of p from $y=px+f(p)$ and $f'(p)+x=0$. Geometrically speaking, this singular solution is the envelope of the family of straight lines $y=Cx+f(C)$, C arbitrary constant, which defines the general solution of (1).

It is therefore natural to expect the appearance of singular solutions for the generalised equation (2). Such solutions are not mentioned in [1]—[3]. In paper [4] and book [5] it is recorded that equation

$$y = xy' - \frac{1}{2}\, x^2\, y'' + (y'')^2 \tag{4}$$

besides the general solution

$$y = Cx + \frac{1}{2}\, Dx^2 + D^2$$

also has the solution $y = \dfrac{1}{48}\, x^4$.

2. We shall first consider a special equation of type (2), obtained by putting $f(t)=t^2$, i.e. the equation

$$y = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!}\, x^k\, y^{(k)} + (y^{(n)})^2. \tag{5}$$

---

* Presented May 17, 1977 by D. S. MITRINOVIĆ.

It is readily verified that the equation (5) is satisfied by any function of the form

$$y = \frac{(-1)^n}{2\,(2\,n)!}\,x^{2\,n} + \sum_{k=1}^{n-1} A_k\,x^k \qquad (A_k \text{ arbitrary constants}). \tag{6}$$

Since the solution (6) is a polynomial of degree $2\,n$, it cannot be obtained from the general solution which is a polynomial of degree n. Hence, (6) is the singular solution of (5).

In particular, the equation (4) has the singular solution $y = \frac{1}{48}\,x^4 + Ax$, where A is an arbitrary constant.

3. For the generalised CLAIRAUT's equation (2) we have:

Equation (2) is satisfied not only by (3), but also by

$$y = Y + \sum_{k=1}^{n-1} A_k\,x^k \qquad (A_k \text{ arbitrary constants}),$$

where Y is any particular solution of the equation

$$f'(y^{(n)}) = \frac{(-1)^n}{n!}\,x^n.$$

4. Certain geometrical relations exist between the general and the singular solution of the generalised CLAIRAUT's equation. We shall exhibit two such properties by considering the equation

$$y = xy' - \frac{1}{2}\,x^2\,y'' + (y'')^2 \tag{4}$$

with the general solution

$$y = Cx + \frac{1}{2}\,Dx^2 + D^2 \qquad (C,\ D \text{ arbitrary constants}), \tag{7}$$

and the singular solution

$$y = \frac{1}{48}\,x^4 + Ax \qquad (A \text{ arbitrary constant}). \tag{8}$$

(i) For each fixed curve E of the family (7) there exist two points P_1 and P_2 and two curves F_1 and F_2 of the form (8) such that E and F_k have a second order contact at $P_k\,(k = 1,\ 2)$. Moreover, it is readily shown that if

$$E\colon\ y = Cx + \frac{1}{2}\,Dx^2 + D^2 \qquad (C,\ D \text{ fixed contants})$$

then

$$P_1 = (2\sqrt{D},\ \ 2\,C\sqrt{D} + 3D^2); \qquad F_1 : \ \frac{1}{48}\,x^4 + \left(\frac{4}{3}\,D\,\sqrt{D} + C\right)x,$$

$$P_2 = (-\,2\sqrt{D},\ \ -2\,C\sqrt{D} + 3\,D^2); \qquad F_2 : \ y = \frac{1}{48}\,x^4 + \left(-\frac{4}{3}\,D\,\sqrt{D} + C\right)x.$$

(ii) For each fixed curve of the family (8) there exist two subfamilies of (7), namely

$$y=\left(\frac{4}{3}D\sqrt{D}+A\right)x+\frac{1}{2}Dx^2+D^2, \qquad y=\left(-\frac{4}{3}D\sqrt{D}+A\right)x+\frac{1}{2}Dx^2+D^2$$

(D arbitrary, A fixed), such that $y=\frac{1}{48}x^4+Ax$ is the envelope of these two families, having in each point a second order contact.

5. We return to equation (5) once again. Put

$$\mathrm{L}_n y=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{(k-1)!}x^{k-1}y^{(k-1)}.$$

Then the equation (5) can be written in the form

$$\mathrm{L}_n y+\frac{1}{(2\,n!)^2}x^{2n}=\left(\frac{(n-1)!}{x^n}\left(\mathrm{L}_n y+\frac{1}{(2\,n!)^2}x^{2n}\right)'\right)^2$$

and it has the obvious solution

$$\mathrm{L}_n y+\frac{1}{(2\,n!)^2}x^{2n}=0,$$

which yields

$$y=\frac{(-1)^n}{2\,(2n)!}x^{2n}+\sum_{k=1}^{n-1}A_k x^k \qquad (A_k \text{ arbitrary constants}),$$

which is precisely the singular solution of (5).

REFERENCES

1. W. H. Witty: *Generalization of Clairaut's differential equation.* Amer. Math. Monthly **59** (1952), 100—102.
2. M. S. Klamkin: *Generalization of Clairaut's differential equation and the analogous difference equation.* Amer. Math. Monthly **60** (1953), 97—99.
3. E. Kamke: *Differentialgleichungen—Lösungsmethoden und Lösungen.* Leipzig 1959.
4. J. D. Kečkić: *Clairaut's equations of higher order.* Publ. Inst. Math. (Beograd) **12** (26) (1971), 63—66.
5. T. L. Saaty: *Modern nonlinear equations.* New York 1967.