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579. NOTES ON CONVEX FUNCTIONS I: A NEW PROOF OF HADAMARD'S INEQUALITIES*

Petar M. Vasić and *Ivan B. Lacković*

1. Let $p>0$ and $q>0$ be given real numbers and let f be a two times differentiable convex function on $[a, b]$ $(-\infty<a<b<+\infty)$. In paper [**1**] the authors of the present paper set the problem of determining the best possible bounds for y so that the inequality

$$f\left(\frac{pa+qb}{p+q}\right)\leqq\frac{1}{2y}\int\limits_{A-y}^{A+y} f(t)\,\mathrm{d}t\leqq\frac{pf(a)+qf(b)}{p+q} \tag{1}$$

holds, where $y\neq 0$ and where $A=\dfrac{pa+qb}{p+q}$. In the case when $p=q=\dfrac{1}{2}$ and $y=\dfrac{b-a}{2}$ inequalities (1) are known under the name HADAMARD's inequalities. In papers [**1**] and [**2**] the following replies are given to the proposed problem:

Theorem A. *For given real numbers $p>0$ and $q>0$ and for an arbitrary two times differentiable convex function f on $[a, b]$, the inequality* (1) *is valid if and only if*

$$y\leqq\frac{b-a}{p+q}\min(p, q)\qquad (y\neq 0). \tag{2}$$

In paper [3] A. LUPAS also solved the above problem. Namely, in paper [3], some properties of the positive linear operators were primarily derived which when applied to the convex function and a suitably chosen positive operator, yield the answer to the set problem. In other words, A. LUPAS in [3] proved theorem A by methods entirely different from those given in [**1**] and [**2**].

In the present paper we shall give a new proof of theorem A, which is, in our opinion, far more natural and simpler than previous ones. In papers [**4**] and [**5**] the method to be used here was already applied but to some other problems, related to convex functions. HADAMARD's inequalities, in form of (1) as far as we know, were not treted in such a way. Owing to the above reasons we quote the results obtained in [**4**] and [**5**].

Namely, in [**4**] K. TODA proved, as early as 1936, the following assertion:

Theorem B. (a) *Each function of the form*

$$\varphi(x)=\sum_{i=0}^{l} p_i\,|x-a_i|+c\qquad (p_i>0) \tag{3}$$

is a convex function.

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(b) *Each convex function f on (a, b) could be approximated by a function of the form* (3).

In paper **[4]** explicit coefficients c, p_i and $a_i \in (a, b)$ are given.

The same assertion was in 1967 rediscovered by T. Popoviciu in paper **[5]**. His theorem, in a somewhat more precise form reads:

Theorem C. (a) *Each function of the sequence*

$$\varphi_m(x) = \lambda x + \mu + \sum_{k=0}^{m} p_k |x - x_k| \quad (m = 1, 2, \dots), \tag{4}$$

where $x \in [a, b]$; $\lambda, \mu \in \mathbf{R}$, $p_k \geqq 0$ $(k = 0, 1, \dots, m)$ and $x_k \in [a, b]$ $(k = 0, 1, \dots, m)$ are arbitrary points, is a convex function on $[a, b]$.

(b) *Each function f convex on $[a, b]$ is the uniform limit of a sequence φ_m of the form* (4) *where $p_k \geqq 0$, $a \leqq x_k \leqq b$ $(k = 0, 1, \dots, m)$ and $\lambda, \mu \in \mathbf{R}$.*

In the above procedure in theorem B the equidistant division, and in theorem C arbitrary division of segment $[a, b]$ was taken into account which does not influence the uniform convergence of the sequence φ_m.

2. In order to determine for which values of $y\,(\neq 0)$ inequality (1) holds, it is necessary and sufficient, on the basis of theorems B and C, to determine for which y inequalities (1) are satisfied for the functions

$$f_1(x) = \lambda x + \mu, \quad f_2(x) = |x - c| \quad (a \leqq x \leqq b) \tag{5}$$

where the point c is arbitrary selected in the segment $[a, b]$.

(a) The function f_1, defined by (5) for arbitrary $\lambda, \mu \in \mathbf{R}$ satisfies inequalities (1) for all $y \neq 0$, $a \leqq A - y < A + y \leqq b$ (inequalities (1) are reduced now to equalities).

(b) Let us investigate further for which $y \neq 0$ inequalities (1) hold for the function f_2 defined by (5), i. e. for which $y \neq 0$

$$\left| \frac{pa + qb}{p + q} - c \right| \leqq \frac{1}{2y} \int_{A-y}^{A+y} |t - c| \,\mathrm{d}t \leqq \frac{p|a-c| + q|b-c|}{p + q} \tag{6}$$

is valied, where $p, q > 0$ are given real numbers, $A = \dfrac{pa + qb}{p + q}$ and where $c \in [a, b]$ is arbitrarly chosen. Owing to the symmetry of the integral appearing in (6) we will assume $y > 0$. We will consider three cases:

1° Let $a \leqq c < A - y$. Then inequalities (6) reduce to

$$A - c \leqq \frac{1}{2y} \, \frac{(A - c + y)^2 - (A - c - y)^2}{2} \leqq \frac{p(c-a) + q(b-c)}{p + q}$$

which also reduces to

$$A - c \leqq A - c \leqq \frac{p(c-a) + q(b-c)}{p + q}. \tag{7}$$

The first inequality in (7) always holds. The second reduces to $p(a - c) \leqq p(c - a)$ which is always true. It means that (6) is valid for every c and $y > 0$ if $a \leqq c < A - y$.

2° Let $A+y<c\leqq b$. In this case we have a situation similar to 1°. Namely, at present inequalities (6) reduce to

$$c-A\leqq\frac{1}{2y}\,\frac{(c-A+y)^2-(c-A-y)^2}{2}\leqq\frac{p(c-a)+q(b-c)}{p+q}$$

i. e.

$$c-A\leqq c-A\leqq\frac{p(c-a)+q(b-c)}{p+q}. \tag{8}$$

The first inequality in (8) is obviously true and the second is reduced to the obviously valid inequality $q(c-b)\leqq q(b-c)$. It means that (6) holds for every c and $y>0$ if $A+y<c\leqq b$.

3° Let $A-y\leqq c\leqq A+y$. Now inequalities (8) get the form

$$|A-c|\leqq\frac{1}{2y}\left(y^2+(A-c)^2\right)\leqq\frac{p(c-a)+q(b-c)}{p+q}. \tag{9}$$

The first inequality in (9) is equivalent to

$$y^2-2y|A-c|+|A-c|^2=(y-|A-c|)^2\geqq 0,$$

which is satisfied for every $y>0$ and $A-y\leqq c\leqq A+y$.

We will consider the second inequality in (9). This inequality is equivalent to

$$c^2-2c(A+\alpha y)+(A^2+y^2-2\beta y)\leqq 0, \tag{10}$$

where

$$\alpha=\frac{p-q}{p+q},\quad \beta=\frac{qb-pa}{p+q}. \tag{11}$$

Let us consider the function F defined by

$$F(c)=c^2-2c(A+\alpha y)+(A^2+y^2-2\beta y)$$

for $c\in[A-y,\,A+y]$. The function F has a minimum for $c_0=A+\alpha y$ for which $A-y<c_0<A+y$. Thus condition $F(c)\leqq 0$ (i. e. inequality (10)) will be satisfied for every $c\in[A-y,\,A+y]$ if and only if

$$F(A-y)\leqq 0 \quad\text{and}\quad F(A+y)\leqq 0. \tag{12}$$

It is directly verified that

$$F(A-y)=2y\big(y(1+\alpha)-A\alpha-\beta\big),\quad F(A+y)=2y\big(y(1-\alpha)-A\alpha-\beta\big). \tag{13}$$

Since $1+\alpha>0$ and $1-\alpha>0$, using (11) and (13) we find that (12) holds if and only if the conditions

$$0<y\leqq\frac{p(b-a)}{p+q}\quad\text{and}\quad 0<y\leqq\frac{q(b-a)}{p+q}.$$

are fulfilled. Thus inequality (6) is valid for $A-y\leqq c\leqq A+y$ if and only if

$$0<y\leqq\frac{b-a}{p+q}\min(p,\,q). \tag{14}$$

Having in view the above, we conclude that inequality (6) holds for every $c\in[a,\,b]$ and $y\neq 0$ if and only if y satisfies (2).

Using the results obtained in points 1°, 2° and 3° we find that inequalities

$$f_1\left(\frac{pa+qb}{p+q}\right)\leqq\frac{1}{2y}\int_{A-y}^{A+y} f_1(t)\,\mathrm{d}t\leqq\frac{pf_1(a)+qf_1(b)}{p+q} \tag{15}$$

and

$$\psi_k\left(\frac{pa+qb}{p+q}\right)\leqq\frac{1}{2y}\int_{A-y}^{A+y} \psi_k(t)\,\mathrm{d}t\leqq\frac{p\psi_k(a)+q\psi_k(b)}{p+q} \tag{16}$$

where $\psi_k(x)=|x-x_k|$ $(a\leqq x_k\leqq b)$ $(k=0, 1, \ldots, m)$ hold for $y\neq 0$ if and only if the condition (2) is fulfilled.

If f is a given convex function on $[a, b]$ then, on the basis of theorem B and C, $p_k\geqq 0$ $(0\leqq k\leqq m)$ holds, so that on the basis of (4), (15) and (16) we have the validity of

$$\varphi_m\left(\frac{pa+qb}{p+q}\right)\leqq\frac{1}{2y}\int_{A-y}^{A+y} \varphi_m(t)\,\mathrm{d}t\leqq\frac{p\varphi_m(a)+q\varphi_m(b)}{p+q} \quad (m=1, 2, \ldots) \tag{17}$$

if and only if condition (2) is fulfilled where $y\neq 0$. As already said sequence φ_m uniformly converges towards function f. On the basis of that from (17) the assertion of theorem A follows.

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