

552. **THERE ARE EXACTLY 13 CONNECTED, CUBIC,
 INTEGRAL GRAPHS***

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1. Results. A graph is called integral if its spectrum consists entirely of integers. Cubic graphs are regular graphs of degree 3.

It was proved in [3] that the set I_r of all connected regular integral graphs of a fixed degree r is finite. At the same time the search for cubic integral graphs was begun. Now we complete this work by the following theorem.

Theorem 1. *There are exactly 13 connected, cubic, integral graphs. They are displayed in Fig. 1 and in Fig. 2 of [3].*

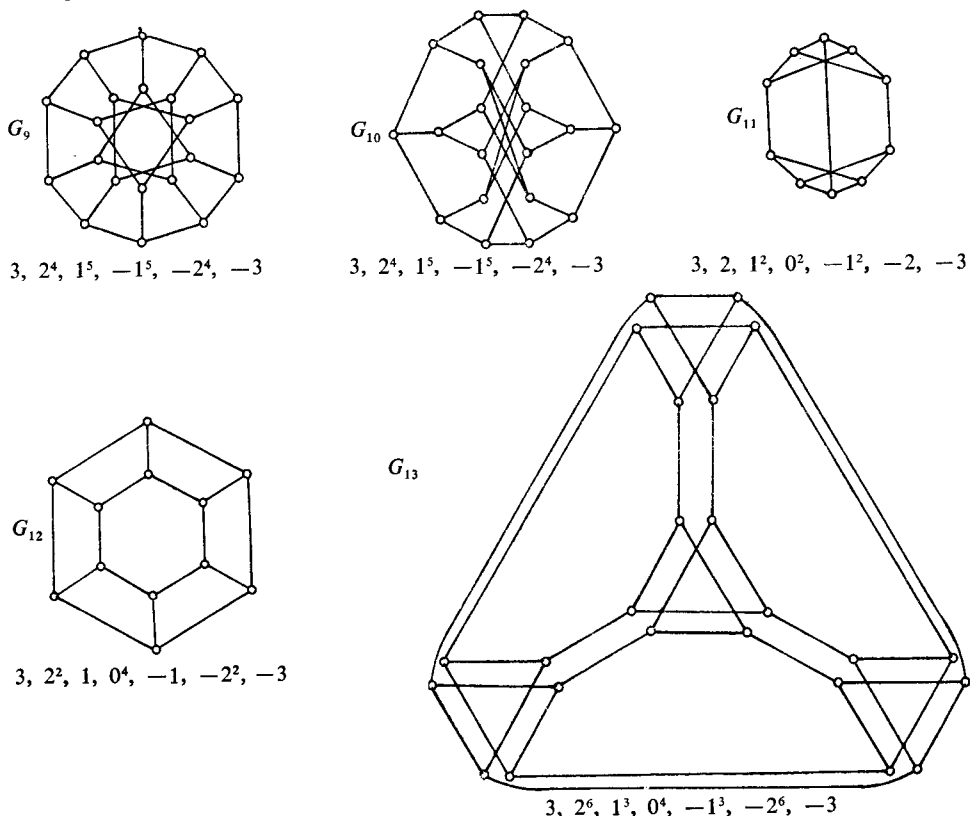


Fig. 1

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Together with the graphs the corresponding spectra are given in Fig. 1. If $\lambda_1, \dots, \lambda_s$ are distinct eigenvalues of a graph with the multiplicities x_1, \dots, x_s , then the whole spectrum is represented by $\lambda_1^{x_1}, \dots, \lambda_s^{x_s}$.

Note that all connected integral graphs, whose vertex degrees do not exceed 3 and are not all equal to 3, are known, c. f. [2]. They are displayed in Fig. 1 of [3]. Combining both facts we have the following theorem.

Theorem 2. *There exist exactly 20 connected integral graphs whose vertex degrees do not exceed 3.*

The proof of Theorem 1 is rather long. It includes the whole paper [3]. We shall give here only an outline of the proof.

2. Six distinct eigenvalues. Cubic integral graphs have, of course, at most seven distinct eigenvalues. All such (connected) graphs with at most five distinct eigenvalues were found in [3]; this are the graphs G_1, \dots, G_8 from Fig. 2 of [3]. If G is a bipartite connected cubic integral graph with six distinct eigenvalues, than G has the spectrum of the Desargues graph (the graph G_9 on Fig. 1), as was proved in [3]. Now we have proved by computer that there is, except for G_9 , exactly one more graph with the same spectrum. It is represented on Fig. 1 as the graph G_{10} .

The graphs G_9 and G_{10} are cospectral but not isomorphic. The order of the automorphisme group of G_9 is 240 and only 48 for G_{10} . But the both graphs have the same number of circuits of length i for any $i=3, 4, \dots, 20$! In the representation by Fig. 2 b) the graphs G_9 and G_{10} differ only by the fact that the edges between V_2 and V_3 form in the first case two hexagons and in the second case a circuit of length 12! Furthermore the numbers of cocliques of the orders 4, 6, 8, 10 in G_9 are 1510, 1320, 115, 2, respectively, and the same numbers for G_{10} are 1510, 1320, 111, 2.

There are some more interesting facts about these two graphs. As it was mentioned in [3] we have the relations $G_9 = G_3 \wedge K_2 = G_7 \wedge K_2$, where \wedge denotes the conjunction (product) of graphs. But G_{10} has not such a decomposition with respect to the conjunction, since in the opposite case a new integral cubic graph on 10 vertices would exist. In addition, this means, that in any representation of the adjacency matrix A of G_{10} in the form

$$(1) \quad A = \begin{vmatrix} 0 & N \\ N^T & 0 \end{vmatrix},$$

where N is a $(0,1)$ -square matrix, the matrix N cannot be symmetric with zero diagonal. But still there is a representation of A such that N is symmetric and satisfies the equation

$$(2) \quad N^2 = 3I + B,$$

where B is the adjacency matrix of the complement of the graph G_3 (Petersen graph). The adjacency matrices of G_3 and G_7 satisfy the equation (2), too! Further we can say that every $(0,1)$ -solution of the equation $NN^T = 3I + B$ by (1) provides either the graph G_9 or the graph G_{10} .

We shall describe briefly the way of finding G_{10} . Using the same procedure as in the case $n=24$ (see below) we established that the girth was equal to 6 and that each vertex is on exactly 6 circuits of length 6. Let us fix one vertex x in the graph (see Fig. 2).

Let V_i denote the set of the vertices which are at distance i from x . We have $|V_0|=1$, $|V_1|=3$, $|V_2|=6$. By consideration of the numbers $|V_i|$, for $i>2$, it turned out that the only possibilities are those of Fig. 2, where in both cases only the edges between V_2 and V_3 are missing. This was sufficient to finish the investigation by a computer search. The first alternative leads to the graph G_{10} and the second one to both G_9 and G_{10} .

For non-bipartite graphs from I_3 with 6 distinct eigenvalues a table of possible spectra was given in [3] (Table 1). Since these graphs have the least eigenvalue equal to -2 , they must be, according to [1], either line graphs or complements to regular graphs of degree 1 (cocktail-party graphs) or they are represented by the so called root system E_8 .

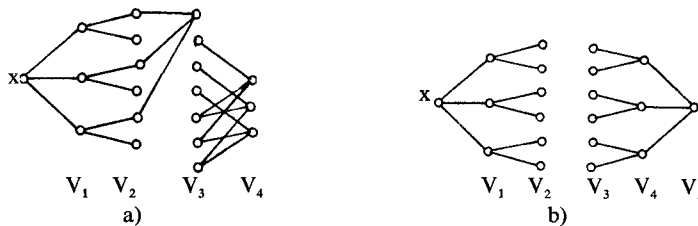


Fig. 2

In the first case it can easily be seen that the graphs must be line graphs of some semi-regular bipartite graphs which are, on the other hand, subdivision graphs of some cubic graphs (possible with multiple edges). According to [4] the line graph of the subdivision graph of a regular connected graph is an integral graph if and only if the starting graph is complete or has two vertices. But in these two cases the number of distinct eigenvalues is not equal to six.

The second case is obviously impossible and in the third case there again is no solution. Indeed, connected cubic graphs which can be represented by E_8 have at most 10 vertices and all such integral graphs have already been mentioned (they have no six distinct eigenvalues).

The non-existence of these graphs can be proved also by calculating the number of circuits of length 4, which we shall carry out for the next class of graphs.

3. Seven distinct eigenvalues. The only (bipartite) graph from I_3 with 7 distinct eigenvalues and with not more than 12 vertices are the graphs G_{11} and G_{12} from Fig. 1, as was mentioned in [3]. The possible spectra of such graphs with more than 12 vertices are given in Table 2 of [3]. It was also noticed that the graph G_{13} from Fig. 1 has a spectrum of Table 2. We shall show now that there are no other such graphs.

If $\lambda_1=r, \lambda_2, \dots, \lambda_n$ are the eigenvalues of a regular graph G of degree r , then the number D_4 of circuits of length 4 (quadrangles) in G is given by

$$D_4 = \frac{1}{8} \left(\sum_{i=1}^n \lambda_i^4 - nr(2r-1) \right),$$

which can easily be checked. Applying this formula on hypothetical graphs from Table 2 of [3], we can establish that they do not exist except for the graphs with the

spectrum of the form $3, 2^x, 1^y, 0^z, -1^y, -2^x, -3$ with the following four sets of parameters:

	n	x	y	z	D_4	D_6
1°	16	3	3	2	3	14
2°	18	4	2	4	3	10
3°	20	5	1	6	3	6
4°	24	6	3	4	0	12

In this table n is the number of vertices and D_6 is the number of circuits of length 6 (hexagons). The number D_6 is calculated by the formula

$$D_6 = -\frac{1}{2}(a_6 + b_3 - 2(m - 8)D_4),$$

where $b_3 = \frac{1}{6}m(m^2 - 15m + 58)$, m is the number of edges and a_6 is the coefficient of λ^{n-6} in the characteristic polynomial $\det(\lambda I - A)$ of G , A being adjacency matrix of G . The above formula holds for cubic graphs of girth 4 and can be derived from the general procedure of calculating the numbers of circuits of certain size in regular graphs [5].

Using HOFFMAN's polynomial we obtain (c.f. [3]) $A^6 + 3A^5 - 5A^4 - 15A^3 + 4A^2 + 12A = \frac{720}{n}J$, where J is the matrix whose all entries are equal to 1. Let σ_k ($k = 1, 2, \dots$) be the number of closed walks of length k which start and terminate at a fixed vertex i . Considering the (i, i) -entries in the last matrix equation we get $\sigma_6 - 5\sigma_4 + 4\sigma_2 = \frac{720}{n}$, since $\sigma_k = 0$ for odd k (the graph is bipartite). Having in mind that our graphs are cubic and bipartite we have $\sigma_2 = 3$ and $\sigma_4 = 15 + 2d_4$, where d_4 is the number of quadrangles containing the vertex i . Hence, we get

$$(3) \quad \sigma_6 = \frac{720}{n} + 63 + 10d_4.$$

On the other hand, if the vertex i does not belong to any quadrangle then σ_6 is at least 87 (what can be seen by inspection) and any hexagon containing i increases that number by 2 and also each circuit of length 4, whose one vertex is adjacent to i , increases that number by 2.

Now the four cases mentioned above are considered separately.

1° $n=16$. We can find a vertex i which is not contained in a circuit of length 4. Therefore $d_4=0$ and we get, from (3), $\sigma_6=108$. But σ_6 must be odd and so the graph does not exist.

2° $n=18$. We take again a vertex i with $\sigma_4=0$ and we get $\sigma_6=103$. Hence, $103-87=16$ of these closed walks contain a quadrangle or a hexagon.

First we shall prove that no 2 of the 3 quadrangles can have a common vertex. Two quadrangles cannot have only one vertex in common since the graph is regular of degree 3. Suppose that they have exactly two common vertices. Then the subgraph from Fig. 3 would appear and there would be 8 vertices in the graph not laying on quadrangles.

The number of closed walks of length 6 starting and terminating at these 8 vertices and containing a quadrangle or a hexagon would be $8 \cdot 16=128$. In order to construct these 128 closed walks we have at our disposal only 9 hexagons since the hexagon contained in the subgraph from Fig. 3 is of no use. These 9 hexagons can provide at most $9 \cdot 12=108$ closed walks of desired type. Quadrangles of the subgraph from

Fig. 3 provide 8 such walks and the third quadrangle provides 8 further such walks. So we have at most $108 + 8 + 8 = 124$ such walks, which is not sufficient. Hence, the subgraph of Fig. 3 is impossible.

Suppose now that two quadrangles have three common vertices, then the subgraph of Fig. 4 would appear.

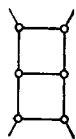


Fig. 3

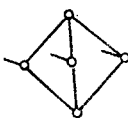


Fig. 4

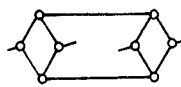


Fig. 5

Now we would have 13 vertices with $d_4=0$ and a similar reasoning as above shows the impossibility of this case.

Hence, the quadrangles are disjoint.

Two quadrangles cannot be joined by more than 1 edge. For example, the situation on Fig. 5 is impossible because this subgraph already contains 4 hexagons and the balance of closed walks for vertices outside quadrangles is not possible any more.

Accordingly, the subgraph induced by vertices of quadrangles can take the form of graphs from Fig. 6.

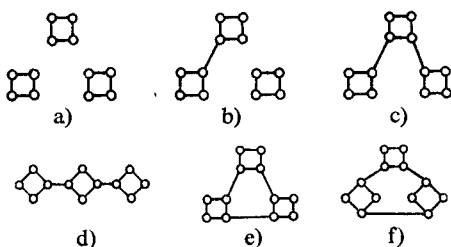


Fig. 6

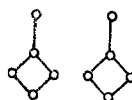


Fig. 7

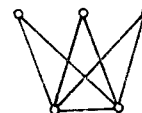


Fig. 8

In the cases b), c), d) one can easily prove that the whole graph contains the subgraph from Fig. 7.

This subgraph has two eigenvalues greater than 2 and this is impossible since the whole graph has only one eigenvalue greater than 2.

In the cases e), f) the graph can be completed in a unique way and one can easily see that the solution does not exist.

So, only the situation a) remains, i. e. two quadrangles cannot be joined by any edge. In this case the subgraph induced by 6 vertices not laying on quadrangles has exactly 3 edges. Now, there are a few variants for completing the graph and it readily follows that in no case we get the desired graph.

Hence, the graph with the spectrum $3, 2^4, 1^2, 0^4, -1^2, -2^4, -3$ does not exist.

3° $n=20$. Since $D_4=3$, there are 8 vertices for which $d_4=0$. For these vertices we have $\sigma_6=99$. So, $99 - 87 = 12$ such closed walks contain a circuit (of length 4 or 6). Further, we have $8 \cdot 12 = 96$ closed walks of this type which start and terminate at one of the mentioned 8 vertices. All these walks are mutually different. On the other hand, each quadrangle can provide 8 such walks and each hexagon provides 12 of them. Hence, the total number is $8 D_4 + 12 D_6 = 96$, which is in agreement with earlier facts.

But, this means that all walks coming out from quadrangles and hexagons must really be taken into account. First, all vertices which are at distance 1 from quadrangles must be contained in the mentioned 8 vertices. It follows from this fact that no two of the three quadrangles can have a common vertex. Further, no vertex of a quadrangle is adjacent to any vertex of other quadrangle.

Consider a quadrangle. Let a, b, c, d be its vertices. Consider the vertices e, f, g, h adjacent to a, b, c, d , respectively, but not laying on that quadrangle. Vertices e, f, g, h are mutually different and mutually nonadjacent.

But also, all closed walks of length 6 which come out of hexagons must really be among the closed walks of length 6 which start in our 8 vertices. This means that all 6 hexagons are contained in the subgraph induced by our 8 vertices which is impossible since this subgraph contains only 6 edges.

4° $n=24$. We have $\sigma_6=93$ since $d_4=0$. Hence, $93-87=6$ closed walks of length 6, starting and terminating in a given vertex, come from hexagons passing through that vertex. This means that each vertex lies in exactly 3 circuits of length 6. This is in agreement with the total number of such circuits.

To prove that the only graph with these properties is the graph G_{13} we needed a long chain of reasoning concerning structural details of the graph. Here we shall only mention the main facts.

As in the case of Desargues graph, take any vertex x and consider the sets V_i of vertices which are at distance i from x . It can be proved that the diameter is 4 and $|V_0|=1$, $|V_1|=3$, $|V_2|=6$, $|V_3|=9$, and $|V_4|=5$. Since the graph is bipartite no pair of vertices from the same set V_i are joined by an edge.

In the next step we established, using the fact that through each vertex there pass exactly 3 hexagons, that the hexagons are only of the following types:

1° 3 hexagons passing through x ;

2° 3 hexagons having 1 vertex in V_1 , 2 vertices in V_2 , 2 vertices in V_3 and 1 vertex in V_4 ;

3° 6 hexagons having 1 vertex in V_2 , 3 vertices in V_3 and 2 vertices in V_4 .

Consider now the vertices from V_4 . Two vertices from V_4 can have at most one common adjacent vertex (from V_3). Define the graph H whose vertex set is V_4 and in which two vertices are adjacent if and only if they have a common adjacent vertex. An important point was to establish that the only possibility for H is the graph on Fig. 8.

Further construction is straightforward and it leads to the graph G_{13} .

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