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549. ADDITIONS TO KAMKE'S TREATISE, VII: VARIATION OF PARAMETERS FOR NONLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS*

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The well known classicial method usually referred to as tbe variation of parameters has been successfully applied to linear differential equations. In the present paper we shall demonstrate that this method can also be applied to certain nonlinear differential equations.

1. Consider the equation

$$y''+f(x)y'=F(x,y,y'). \tag{1.1}$$

We first solve the linear part of (1.1), namely the equation

$$y''+f(x)y'=0. \tag{1.2}$$

From (1.2) follows

$$y'=Ke^{-\int f(x)\,\mathrm{d}x}, \tag{1.3}$$

where K is an arbitrary constant. Suppose that K is a differentiable function of y, and differentiate (1.3) to obtain

$$y''=K'(y)K(y)e^{-2\int f(x)\,\mathrm{d}x}-K(y)f(x)e^{-\int f(x)\,\mathrm{d}x}. \tag{1.4}$$

Substituting (1.3) and (1.4) into (1.1) we find

$$K'(y)K(y)e^{-2\int f(x)\,\mathrm{d}x}=F\Big(x,\ y,\ K(y)e^{-\int f(x)\,\mathrm{d}x}\Big). \tag{1.5}$$

Clearly, (1.5) will be a first order differential equation for $K(y)$ if

$$F\Big(x,\ y,\ K(y)\mathrm{e}^{-\int f(x)\,\mathrm{d}x}\Big)=e^{-2\int f(x)\,\mathrm{d}x}M\big(y,\ K(y)\big). \tag{1.6}$$

Equality (1.6) will take place if, for example,

$$F(u,v,w)=\sum_{\nu=1}^{n}h_\nu(v)\,w^{\alpha_\nu}e^{(\alpha_\nu-2)\int f(u)\,du},$$

where h_ν are arbitrary functions, α_ν are real numbers, and n is a positive integer. In that case (1.5) becomes

$$K'(y)=\sum_{\nu=1}^{n}h_\nu(y)K(y)^{\alpha_\nu-1}. \tag{1.7}$$

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We therefore arrive at the following result:

The second order differential equation

(1.8) $$y''+f(x)\,y'=\sum_{\nu=1}^{n} h_\nu(y)\,(y')^{\alpha_\nu}\,e^{(\alpha_\nu-2)\int f(x)\,\mathrm{d}x}$$

can be reduced to the first order equation (1.7).

Furthermore, if $K(y)=\Phi(y, A)$, where A is an arbitrary constant, is the general solution of (1.7), then the general solution of (1.8) is given by

$$\int \frac{\mathrm{d}y}{\Phi(y, A)}=\int e^{-\int f(x)\,\mathrm{d}x}\,\mathrm{d}x+B,$$

where B is the other arbitrary constant.

2. We now consider some particular cases of equations (1.7) and (1.8).

(i) It is known (see, for example, **[1]**) that the following two generalized forms of EMDEN's equation

$$y''+f(x)\,y'+g(x)\,y^r=0 \qquad (r \text{ real number})$$

and

$$y''+f(x)\,y'+g(x)\,e^y=0$$

are integrable by quadratures if $g(x)=ce^{-2\int f(x)\,\mathrm{d}x}$ ($c=$const). The same result holds for the equation

(2.1) $$y''+f(x)\,y'+g(x)\,h(y)=0,$$

with $g(x)=ce^{-2\int f(x)\,\mathrm{d}x}$.

Indeed, equation (2.1) is obtained from (1.8) by setting $n=1$, $\alpha_1=0$, $h_1(y)=-ch(y)$. The corresponding equation (1.7) takes the following simple form: $K'(y)\,K(y)+ch(y)=0$.

REMARK. This result was published in **[2]**.

(ii) Let $n=2$, $\alpha_1=2$, $\alpha_2=\alpha$. Equation (1.7) becomes a BERNOULLI type equation

$$K'(y)=h_1(y)\,K(y)+h_2(y)\,K(y)^{\alpha-1},$$

and hence the corresponding equation (1.8)

(2.2) $$y''+f(x)\,y'=h_1(y)\,(y')^2+h_2(y)\,(y')^\alpha\,e^{(\alpha-2)\int f(x)\,\mathrm{d}x}$$

can be integrated by quadratures.

In particular, for $\alpha=0$ we have the equation

(2.3) $$y''+f(x)\,y'=h_1(y)\,(y')^2+h_2(y)\,e^{-2\int f(x)\,\mathrm{d}x}.$$

Equation (2.3) is in connection with an equation given by R. T. HERBST **[3]** (see also W. F. AMES **[4]**, p. 62). Namely, HERBST proved the result:

The nonlinear equation

(2.4) $$y''+f(x)\,y'+q(x)\,Z(y)=A(y)\,(y')^2+C(y)\,e^{-2\int f(x)\,\mathrm{d}x},$$

where

(2.5) $$ZC' + (3 - AZ)C = 0, \qquad Z' - AZ = 1$$

has general solution $y = F(u, v)$, where u, v are independent solutions of the linear equation

$$Y'' + f(x)Y' + q(x)Y = 0.$$

For $q(x) \equiv 0$, HERBST's equation (2.4) becomes (2.3). However, the fact that equation (2.3) is integrable by quadratures is not a consequence of HERBST's result, since the functions $A(y)$ and $C(y)$, which appear in (2.4) for $q = 0$ are tied by (2.5), or eliminating $Z(y)$, by

$$\left(\frac{3C}{AC - C'}\right)' - \frac{3AC}{AC - C'} = 1,$$

while the functions $h_1(y)$ and $h_2(y)$ in (2.3) are arbitrary.

Hence, the class of differential equations defined by (2.3) overlaps with the class defined by (2.4) and (2.5).

(iii) Let $n = 3$, $\alpha_1 = 0$, $\alpha_2 = 2$, $\alpha_3 = m$, $h_1(y) = -a\psi(y)$, $h_2(y) = -\varphi(y)$, $h_3(y) = -b\beta(y)$. We obtain the equation

(2.6) $$y'' + f(x)y' + a\psi(y)e^{-2\int f(x)\,dx} + \varphi(y)(y')^2 + b\beta(y)(y')^m e^{(m-2)\int f(x)\,dx} = 0,$$

considered by L. M. BERKOVIČ and N. H. ROZOV [5], who reduced it to the equation

(2.7) $$\frac{d^2 z}{dt^2} + a\psi(z) + \varphi(z)\left(\frac{dz}{dt}\right)^2 + b\beta(z)\left(\frac{dz}{dt}\right)^m = 0.$$

According to our result, equation (2.6) can be reduced to the equation

(2.8) $$K'(y) + a\psi(y)K(y)^{-1} + \varphi(y)K(y) + b\beta(y)K(y)^{m-1} = 0.$$

However, equation (2.7) after the substitution $\frac{dz}{dt} = K(y)$, becomes (2.8) which means that the result of BERKOVIČ and ROZOV is a special case of our result.

(iv) For $n = 3$, $\alpha_\nu = \nu$, equation (1.7) becomes the RICCATI equation

(2.9) $$K'(y) = \sum_{\nu=1}^{3} h_\nu(y)K(y)^{\nu-1}.$$

There are many conditions which ensure the integrability of equation (2.9). A number of those is given in KAMKE [6]. Hence, each of those conditions implies the integrability of the equation

(2.10) $$y'' + f(x)y' = h_1(y)y'e^{-\int f(x)\,dx} + h_2(y)(y')^2 + h_3(y)(y')^3 e^{\int f(x)\,dx}.$$

It is also interesting to note the following fact.

Equation (2.9) is equivalent to the linear equation

(2.11) $$h_3(y)K''(y) - \big(h_3'(y) + h_3(y)h_2(y)\big)K'(y) + h_3(y)^2 h_1(y)K(y) = 0.$$

Hence, the nonlinear equation (2.10) can be reduced to the linear equation (2.11).

(v) For $n=4$, $\alpha_\nu=\nu$, equation (1.7) becomes the ABEL equation of the first kind

$$K'(y)=\sum_{\nu=1}^{4} h_\nu(y)\,K(y)^{\nu-1}.$$

A number of integrable ABEL's equations is recorded in **[6]**. Each of these cases leads to an integrable second order equation

$$y''+f(x)\,y'=\sum_{\nu=1}^{4} h_\nu(y)\,(y')^\nu\,e^{(\nu-2)\int f(x)\,dx}.$$

3. There exist equations which can be reduced to equation (1.8). We exhibit one such example.

Consider the equation

(3.1) $$y''+f(x)\,y'+q(x)\,y=F(x)\,y^k.$$

Put $y=pu$, where p satisfies the linear equation

(3.2) $$p''+f(x)p'+q(x)p=0.$$

We find

(3.3) $$u''+\left(2\frac{p'}{p}+f\right)u'=F(x)\,p^{k-1}\,u^k.$$

Equation (3.3) will have the form (1.8) if

$$F(x)=cp^{-k-3}\,e^{-2\int f(x)\,dx}\qquad(c=\text{const}).$$

Hence, the differential equation

(3.4) $$y''+f(x)\,y'+q(x)\,y=cp(x)^{-k-3}\,e^{-2\int f(x)\,dx}\,y^k\qquad(c=\text{const})$$

where p satisfies (3.2) is integrable by quadratures.

In particular, for $f(x)\equiv 0$, we obtain the integrable equation

(3.5) $$y''+q(x)\,y=cp(x)^{-k-3}\,y^k\qquad(c=\text{const}),$$

where p satisfies (3.2).

In the special case $k=-3$, equation (3.5) reduces to the equation

(3.6) $$y''+q(x)\,y=cy^{-3}\qquad(c=\text{const})$$

which was solved, as claimed by L. M. BERKOVIČ and N. H. ROZOV **[7]**, by V. P. ERMAKOV in 1880. Nevertheless, it was a short note by E. PINNEY **[8]** regarding (3.6) that served as a starting point for a fruitful search of exact solutions of nonlinear second and higher order equations (see, for example, **[3]** and **[9]** — **[14]**).

It is interesting to note that BERKOVIČ and ROZOV solved in the mentioned paper **[7]** equation (3.5) but under the condition that p is a solution of ERMAKOV-PINNEY's equation (3.6).

4. We now apply the variation of parameters to an other second order equation.

Consider the equation

$$y''+g(y)(y')^2=G(x, y, y').$$

We first obtain a first integral of the equation

$$y''+g(y)(y')^2=0$$

in the form

$$y'=Ke^{-\int g(y)\,dy},$$

where K is an arbitrary constant.

Supposing that K is a differentiable function of x, and applying the procedure similar to that of section 1, we arrive at the following result:

The second order differential equation

$$(4.1) \qquad y''+g(y)(y')^2=\sum_{\nu=0}^{n} h_\nu(x)(y')^{\alpha_\nu} e^{(\alpha_\nu-1)\int g(y)\,dy}$$

can be reduced to the first order equation

$$(4.2) \qquad K'(x)=\sum_{\nu=1}^{n} h_\nu(x)\,K(x)^{\alpha_\nu}.$$

Furthermore, if $K(x)=\psi(x, A)$, where A is an arbitrary constant, is the general solution of (4.2), then the general solution of (4.1) is given by

$$\int e^{\int g(y)\,dy}dy=B+\int\psi(x, A)\,dx,$$

where B is the other arbitrary constant.

5. Some interesting particular cases of (4.1) can be formed. As an example we mention the equation

$$(5.1) \qquad y''-\frac{f'(y)}{f(y)}(y')^2+F(x)\,y'+H(x)\,f(y)=0$$

recorded by KAMKE [6] as equation 6.52.

Equation (5.1) is obtained from (4.1) for $n=2$, $\alpha_1=1$, $\alpha_2=0$, $g(y)=-\dfrac{f'(y)}{f(y)}$, $h_1(x)=-F(x)$, $h_2(x)=-H(y)$. The corresponding equation (4.2) becomes in that case the linear equation

$$(5.2) \qquad K'(x)+F(x)\,K(x)+H(x)=0$$

and hence (5.1) can be integrated by quadratures.

In particular, for $f(y)=y$, we obtain the equation

$$(5.3) \qquad y''+F(x)\,y'-y^{-1}(y')^2+H(x)\,y=0.$$

P. PAINLEVÉ showed in [15] (see also [4], p. 60 and [6], equation 6.129) that the solution of the equation

$$(5.4) \qquad y''+F(x)\,y'+ay^{-1}(y')^2+H(x)\,y=0$$

is $y=u^{1/(1+a)}$, where u is the solution of

$$u''+F(x)\,u'+(a+1)\,H(x)\,u=0.$$

Painlevé's result clearly does not hold for $a=-1$. However, for $a=-1$, equation (5.4) becomes (5.3) with the corresponding equation (5.2) and can therefore be integrated by quadratures.

Again, for $n=2$, $\alpha_1=0$, $\alpha_2=\alpha$, we obtain the equation

(5.5) $$y''+g(y)(y')^2=h_1(x)\,e^{-\int g(y)\,dy}+h_2(x)(y')^{\alpha}\,e^{(\alpha-1)\int g(y)\,dy}$$

which is integrable by quadratures because the corresponding first order equation is a Bernoulli type equation

$$K'(x)=h_1(x)\,K(x)+h_2(x)\,K(x)^{\alpha}.$$

In the special case, when $g(y)=ay^{-1}$ ($a=\text{const}\neq 0$), (5.5) becomes

(5.6) $$y''+ay^{-1}(y')^2=h_1(x)\,y^{-a}+h_2(x)(y')^{\alpha}\,y^{a(\alpha-1)}.$$

A substantial number of equations of type (5.6) is recorded by Kamke [6].

6. It might be of interest to note that the equation

(6.1) $$y''+f(x)\,y'+g(y)(y')^2=0,$$

often referred to as Liouville's equation, can be integrated by both methods given in section 1 and section 4. (It can also be integrated directly after dividing by y'.)

The reason is that equation (6.1) does not change its form if x is taken as the independent variable of the dependent variable y.

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