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546. A NOTE ON THE MINIMUM VALUE OF A DEFINITE INTEGRAL*

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0. In the book [1] by F. BOWMAN and F. A. GERARD the following problem has been raised:

Prove that the minimum value of the integral

$$\int_0^{+\infty} e^{-x}(1+a_1 x+\cdots+a_n x^n)^2\,\mathrm{d}x$$

is equal $\dfrac{1}{n+1}$.

In [2] L. J. MORDELL has given a simple and ellegant proof of this result. Subsequently F. SMITHIES [3] has proved the same result using the orthogonality properties of the LAGUERRE polynomials. At the same time he has derived the explicit expression for the minimizing polynomial. In [4] L. J. MORDELL has solved, in some cases, the problem of finding the minimum value of integrals of the form

$$(1)\qquad \int_a^b p(x)\,f_n(x)^2\,\mathrm{d}x=\int_a^b p(x)\Big(\sum_{i=0}^{n} b_i x^i\Big)^2\mathrm{d}x \qquad (b_i\in\mathbf{R}),$$

where $p:[a, b]\to\mathbf{R}_+$ is such that the integrals $\int_a^b p(x)\,x^r\,\mathrm{d}x$ $(r\geqq 0)$ exist and the coefficient b_k of the term $b_k x^k$ in the bracket is given as 1. Also, L. MIRSKY [5] has found the minimum of the integral

$$\int_a^b p(x)\,(x^{k_0}+\lambda_1 x^{k_1}+\cdots+\lambda_n x^{k_n})^2\,\mathrm{d}x.$$

His method is of interest since it illustrates the effective use of a simple principle of linear algebra in certain questions of analysis.

In this note we shall find the minimum value of (1) by using a different kind of normalization of $f_n(x)$ which is especially appropriate in solving the approximation problem in the synthesis of filtering networks in communication engineering. Namely, we can fix the value of f_n at $x=c\,(a\leqq c\leqq b)$ so that the integral (1) is to be minimized under the constraint $\sum_{i=0}^{n} b_i c^i=1$. The method presented in this paper is based on the properties of orthogonal polynomials but is different from that reported in [3].

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1. Let p be an arbitrary continuous nonnegative function over finite or infinite interval $[a, b]$ such that $\int_a^b p(x) f(x)^2 \, dx$ exist, where f is an arbitrary polynomial in x. Then a set of polynomials $Q_0, Q_1, Q_2, \ldots$ can be determined that are orthonormal with respect to the weight function p (see [6]).

In order to find the minimum M of the integral

$$\int_a^b p(x) f_n(x)^2 \, dx = \int_a^b p(x) \Big(\sum_{i=0}^n b_i x^i \Big)^2 dx \tag{2}$$

under the constraint $\sum_{i=0}^n b_i c^i = 1$, we expand f_n into a series of Q_i and consider the associated function

$$\varphi(a_0, a_1, \ldots, a_n, \beta) = \int_a^b p(x) \Big(\sum_{i=0}^n a_i Q_i(x) \Big)^2 dx + \beta \Big(\sum_{i=0}^n a_i Q_i(c) - 1 \Big).$$

The necessary conditions for the minimum of the integral (2) are

$$\frac{\partial \varphi}{\partial a_i} \equiv 2 \int_a^b p(x) a_i Q_i(x)^2 \, dx + \beta Q_i(c) = 0 \qquad (i = 0, 1, \ldots, n),$$

$$\sum_{i=0}^n a_i Q_i(c) = 1,$$

wherefrom

$$a_i = \frac{Q_i(c)}{Q_0(c)} a_0, \qquad a_0 \Big(\sum_{i=0}^n Q_i(c)^2 \Big) = Q_0(c),$$

and, hence,

$$a_i = \frac{Q_i(c)}{\sum_{i=0}^n Q_i(c)^2}.$$

Since the existence of a minimum usually stems from the nature of the physical problem under consideration we can write

$$M = \frac{1}{\sum_{i=0}^n Q_i(c)^2} \tag{3}$$

where $Q_0, Q_1, Q_2, \ldots$ represents a set of orthonormal polynomials associated with the weight function p.

2. *Special cases.* If $p(x) = (1 - x^2)^{\lambda - 1/2}$ $\left(\lambda > -\frac{1}{2}\right)$ and $a = -1$, $b = 1$, $c = 1$, the orthonormal set of polynomials reduces to

$$Q_i(x) = \frac{C_i^\lambda(x)}{\sqrt{h_i}} \qquad (i = 0, 1, 2, \ldots),$$

where C_i^λ is the GEGENBAUER polynomial and

$$h_i=\frac{\pi 2^{1-2\lambda}\Gamma(i+2\lambda)}{i!(i+\lambda)\Gamma(\lambda)^2}\qquad \left(\lambda>-\frac{1}{2},\ \lambda\neq 0,\ i=0,1,2,\dots\right),$$

$$=\frac{2\pi}{n^2}\qquad (\lambda=0;\ i=0,1,2,\dots).$$

Now, from (3) we find

$$M=\frac{1}{\sum_{i=0}^{n}\frac{C_i^\lambda(1)^2}{h_i}}=\frac{1}{\sum_{i=0}^{n}\frac{\Gamma(i+2\lambda)(i+\lambda)\Gamma(\lambda)^2}{i!\,\Gamma(2\lambda)^2\pi 2^{1-2\lambda}}}.$$

But

(4) $$\sum_{i=0}^{n}\frac{\Gamma(i+2\lambda)}{i!\,\Gamma(2\lambda)^2}\frac{(i+\lambda)\Gamma(\lambda)^2}{\pi 2^{1-2\lambda}}=\frac{\lambda(2\lambda)_{n+1}(2\lambda+2n+1)\Gamma(\lambda)^2}{2^{1-2\lambda}\pi n!\,\Gamma(2\lambda+2)},$$

so that, by substituting $\lambda-\frac{1}{2}=\alpha$, we finally have for $\sum_{i=0}^{n} b_i=1$,

(5) $$\int_{-1}^{+1}(1-x^2)^\alpha\left(\sum_{i=0}^{n} b_i x^i\right)^2 dx\geq\frac{\pi n!\,\Gamma(2\alpha+3)}{(2\alpha+2)_n\Gamma\left(\alpha+\frac{3}{2}\right)^2(\alpha+n+1)2^{2+2\alpha}}.$$

It can easily be verified that (5) is valid for all $\alpha>-1$, including $\alpha=-\frac{1}{2}$.

The identity (4) is the consequence of the following result:

$$\sum_{i=0}^{n}\frac{\Gamma(i+2\lambda)}{i!\,\Gamma(2\lambda)}(i+\lambda)=\frac{(2\lambda)_{n+1}(2\lambda+2n+1)}{2n!\,(2\lambda+1)}.$$

If λ is an nonnegative integer the above identity can be derived by the method proposed by R. R. JANIĆ [7]

$$\sum_{i=0}^{n}\frac{\Gamma(i+2\lambda)}{i!\,\Gamma(2\lambda)}(i+\lambda)=\sum_{i=0}^{n}\binom{i+2\lambda-1}{2\lambda-1}(i+\lambda)$$

$$=-\lambda\sum_{i=0}^{n}\binom{i+2\lambda-1}{2\lambda-1}+2\lambda\sum_{i=0}^{n}\binom{i+2\lambda}{2\lambda}$$

$$=\lambda\left(2\binom{n+2\lambda+1}{2\lambda+1}-\binom{n+2\lambda}{2\lambda}\right)$$

$$=\frac{(2\lambda)_{n+1}(2\lambda+2n+1)}{2n!\,(2\lambda+1)},$$

where the following relation has been taken into account [8]:

$$\sum_{i=0}^{n}\binom{\nu+i}{\nu}=\binom{n+1+\nu}{\nu+1}.$$

The identity (4) is also walid for noninteger values of λ $\left(\lambda > -\frac{1}{2}\right)$ which can be proved by induction. For $n=0$ the result is obviously corect. Assuming that (4) is also valid for any positive integer value of n, we have,

$$\sum_{i=0}^{n+1} \frac{\Gamma(i+2\lambda)}{i!\,\Gamma(2\lambda)}(i+\lambda) = \frac{(2\lambda)_{n+1}(2\lambda+2n+1)}{2n!\,(2\lambda+1)} + \frac{\Gamma(n+1+2\lambda)}{(n+1)!\,\Gamma(2\lambda)}(n+1+\lambda)$$
$$= \frac{(2\lambda)_{n+2}(2\lambda+2n+3)}{2(n+1)!\,(2\lambda+1)},$$

which completes the proof.

The relation (5) for $\alpha=0$ reduces to

$$\int_{-1}^{+1} \left(\sum_{i=0}^{n} b_i x^i\right)^2 \mathrm{d}x \geqq \frac{2}{(n+1)^2} \qquad \left(\sum_{i=0}^{n} b_i = 1\right),$$

and, for $\alpha = -\frac{1}{2}$,

$$\int_{-1}^{+1} (1-x^2)^{-1/2} \left(\sum_{i=0}^{n} b_i x^i\right)^2 \mathrm{d}x \geqq \frac{\pi}{2n+1} \qquad \left(\sum_{i=0}^{n} b_i = 1\right).$$

In communication engineering the case when $\sum_{i=0}^{n} b_i x^i$ is an even or and odd polynomial for n even or odd respectively is of particular interest since the characteristic function of all-pole filters must be an even or an odd function of frequency depending on whether n is even or odd. In this case, using the same constraint as before $\sum_{i=0}^{n} b_i = 1$, we easily find for $\alpha > -1$

$$\int_{-1}^{+1} (1-x^2)^{\alpha} \left(\sum_{i=0}^{[n/2]} b_{n-2i} x^{n-2i}\right)^2 \mathrm{d}x \geqq \frac{\pi(\alpha+1)\,n!\,\Gamma(2\alpha+2)^2}{2^{2\alpha}\,\Gamma\left(\alpha+\frac{3}{2}\right)^2 \Gamma(n+2\alpha+3)}.$$

The last result has been derived by man use of the following summation formulas

$$\sum_{i=0}^{[n/2]} \binom{n+2\lambda-2i-1}{n-2i} (n+\lambda-2i) = \lambda\, C_n^{\lambda+1}(1).$$

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