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545. SOME NEW SUMS FOR ORTHOGONAL POLYNOMIALS INVOLVED IN FILTER SYNTHESIS*

Branko D. Rakovich and *Petar M. Vasić*

0. Before synthesizing filtering networks which find a widespread use in communication and electronic engineering it is necessary to express the required response as a rational function of the complex frequency. Of course, this function must fulfil the realizability conditions imposed by the nature of physical elements used to implement the network. An important aspect of this part of filter synthesis, which is usually referred to as "the approximation problem", is that there is no unique solution, no "best" approximation but the choice of the approximation norm largely depends on the intended application of the resulting filter. If the magnitude response of the filter is of overriding importance the approximation error is frequently stated in terms of a CHEBYSHEV norm but also the least mean square error criterion can be used to minimize the passband loss of the filter. In these instances CHEBYSHEV, LEGENDRE and other classes of classical orthogonal polynomials are employed to derive the characteristic function of the filter. In what follows some relationships involved in this derivation will be presented.

1. We shall start from the well known recurrence relation for the GEGENBAUER polynomials

(1.1) $$(n+\lambda+1)\, C_{n+1}^{\lambda}(x)=\lambda\left(C_{n+1}^{\lambda+1}(x)-C_{n-1}^{\lambda+1}(x)\right).$$

Substituting $n=2i-1$ $(i=1, 2, \ldots, k)$ in (1.1) and summing the expressions obtained we find

$$\begin{aligned}\sum_{i=1}^{k}(2i+\lambda)\, C_{2i}^{\lambda}(x)&=\lambda\left(\sum_{i=1}^{k}\left(C_{2i}^{\lambda+1}(x)-C_{2i-2}^{\lambda+1}(x)\right)\right)\\&=\lambda\left(\sum_{i=1}^{k}C_{2i}^{\lambda+1}(x)-\sum_{i=0}^{k-1}C_{2i}^{\lambda+1}(x)\right)\\&=\lambda\left(C_{2k}^{\lambda+1}(x)-C_{0}^{\lambda+1}(x)\right)\end{aligned}$$

i.e.

(1.2) $$\sum_{i=0}^{k}(2i+\lambda)\, C_{2i}^{\lambda}(x)=\lambda\, C_{2k}^{\lambda+1}(x).$$

Similarly, substituting $n=2i$ $(i=1, 2, \ldots, k)$ in (1.1) we obtain after summation

(1.3) $$\sum_{i=0}^{k}(2i+1+\lambda)\, C_{2i+1}^{\lambda}(x)=\lambda\, C_{2k+1}^{\lambda+1}(x).$$

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The sums (1.2) and (1.3) can be written in compact form

$$(1.4)\qquad \sum_{r=0}^{\left[\frac{n}{2}\right]} (n-2r+\lambda)\, C_{n-2r}^{\lambda}(x) = \lambda\, C_n^{\lambda+1}(x).$$

Now, for $\lambda = 1/2$, and since $P_n(x) = C_n^{1/2}(x)$, $\frac{d}{dx} P_{n+1}(x) = C_n^{3/2}(x)$, where $P_n(x)$ is the LEGENDRE polynomial, from (1.4) we deduce the well-known result for the LEGENDRE polynomials (see, for instance, [1], p. 20)

$$(1.5)\qquad \sum_{r=0}^{\left[\frac{n}{2}\right]} (2n-4r+1)\, P_{n-2r}(x) = \frac{dP_{n+1}(x)}{dx}.$$

If $\lambda \neq 0$, (1.2) can be written in the form

$$(1.6)\qquad 1 + \sum_{i=1}^{k} (2i+\lambda)\frac{C_{2i}^{\lambda}(x)}{\lambda} = C_{2k}^{\lambda+1}(x).$$

Since

$$\lim_{\lambda\to 0} \frac{C_n^{\lambda}(x)}{\lambda} = \frac{2}{n}\, T_n(x)$$

where $T\ (x)$ is the CHEBYSHEV polynomial of the first kind, and the following results are recovered from (1.3) and (1.6)

$$(1.7)\qquad \begin{aligned} &\sum_{i=1}^{k} T_{2i}(x) = \frac{1}{2}\left(U_{2k}(x) - 1\right), \\ &\sum_{i=0}^{k} T_{2i+1}(x) = \frac{1}{2}\, U_{2k+1}(x), \end{aligned}$$

where $U_n(x)$ is the CHEBYSHEV polynomial of the second kind.

In terms of hypergeometric functions, the sum (1.2) takes the form

$$(1.8)\qquad \lambda + \sum_{i=1}^{k} (2i+\lambda)\frac{\Gamma(2i+2\lambda)}{(2i)!\,\Gamma(2\lambda)}\, {}_2F_1\left(-2i,\, 2i+2\lambda;\ \lambda+\frac{1}{2};\ \frac{1-x}{2}\right) = \lambda\, C_{2k}^{\lambda+1}(x)$$

in which ${}_2F_1$ is undefined for $\lambda = -\frac{1}{2}$. Now dividing both sides of (1.8) by

$$\lambda(2\lambda+1)\,\Gamma\left(\frac{2\lambda+1}{2}\right)(\neq 0)$$

we get

$$(1.9)\qquad 2\sum_{i=1}^{k}(2i+\lambda)\frac{\Gamma(2i+2\lambda)}{(2i)!\,\Gamma(2\lambda+2)\,\Gamma\left(\frac{2\lambda+1}{2}\right)}\, {}_2F_1\left(-2i,\, 2i+2\lambda;\ \lambda+\frac{1}{2},\ \frac{1-x}{2}\right)$$

$$+ \frac{1}{2\,\Gamma\left(\frac{2\lambda+3}{2}\right)} = \frac{C_{2k}^{\lambda+1}(x)}{2\,\Gamma\left(\frac{2\lambda+3}{2}\right)}.$$

For $c=0$, the hypergeometric function ${}_2F_1(a, b; c; z)$ can be transformed into the form

$$\lim_{c\to 0} \frac{{}_2F_1(a, b; c; z)}{\Gamma(c)} = abz\, {}_2F_1(a+1, b+1; 2; z) \tag{1.10}$$

so that for $\lambda \to -1/2$ we have from (1.9) and (1.10)

$$1+2\sum_{i=1}^{k}\left(2i-\frac{1}{2}\right)(x-1)\,{}_2F_1\left(-(2i-1),\, 2i;\, 2;\, \frac{1-x}{2}\right) = P_{2k}(x). \tag{1.11}$$

Also, in a similar way, starting from (1.3) we arrive at

$$x+2\sum_{i=1}^{k}\left(2i+\frac{1}{2}\right)(x-1)\,{}_2F_1\left(-2i,\, 2i+1;\, 2;\, \frac{1-x}{2}\right) = P_{2k+1}(x). \tag{1.12}$$

In order to derive the expression for $\lambda=-1$ we start from (1.8) which can be written in the form

$$\sum_{i=2}^{k}(2i+\lambda)\frac{\Gamma(2\lambda+2i)}{(2i)!\,\Gamma(2\lambda)}\,{}_2F_1\left(-2i,\, 2i+2\lambda,\, \lambda+\frac{1}{2};\, \frac{1-x}{2}\right)$$
$$=\lambda C_{2k}^{\lambda+1}(x)-(\lambda+2)\,C_2^{\lambda}(x)-\lambda\, C_0^{\lambda}(x)$$

and rearranging the terms of both sides

$$4\lambda(2\lambda+1)\sum_{i=2}^{k}\frac{(2i+\lambda)(2\lambda+3)_{2i-3}}{(2i)!}\,{}_2F_1\left(-2i,\, 2i+2\lambda;\, \lambda+\frac{1}{2},\, \frac{1-x}{2}\right)$$
$$=\frac{\lambda\, C_{2k}^{\lambda+1}(x)}{\lambda+1}-2\lambda(\lambda+2)\,x^2+\lambda,$$

which for $\lambda\to -1$ reduces to

$$\sum_{i=2}^{k}\frac{1}{i(i-1)}\,{}_2F_1\left(-2i,\, 2i-2,\, -\frac{1}{2};\, \frac{1-x}{2}\right) = T_2(x)-\frac{1}{k}\,T_{2k}(x). \tag{1.13}$$

Also, starting from (1.3), we find

$$\sum_{i=1}^{k}\frac{2}{4i^2-1}\,{}_2F_1\left(-2i-1,\, 2i-1;\, -\frac{1}{2};\, \frac{1-x}{2}\right) = T_1(x)-\frac{1}{2k+1}\,T_{2k+1}(x), \tag{1.14}$$

so that it follows from (1.13) and (1.14) for all n

$$\frac{1}{n-1}\,T_{n-1}(x)-\frac{1}{n+1}\,T_{n+1}(x)=\frac{2}{n^2-1}\,{}_2F_1\left(-n-1,\, n-1,\, -\frac{1}{2},\, \frac{1-x}{2}\right). \tag{1.15}$$

Combining (1.2) and (1.3) we get

$$\sum_{i=0}^{n}(i+\lambda)\,C_i^{\lambda}(x)=\lambda\left(C_n^{\lambda+1}(x)+C_{n-1}^{\lambda+1}(x)\right) \tag{1.16}$$

and since, after an obvious error correction in [2]

$$\sum_{i=0}^{n}(i+\lambda)\,C_i^{\lambda}(x)=\frac{1}{2}\,\frac{(2\lambda)_{n+1}}{\left(\lambda+\frac{1}{2}\right)_n}\,P_n^{\lambda+1/2,\,\lambda-1/2}(x), \tag{1.17}$$

where $P_n^{\alpha,\beta}(x)$ is the JACOBI polynomial, we have from (1.16) and (1.17)

$$C_n^{\lambda+1}(x)+C_{n-1}^{\lambda+1}(x)=\frac{(2\lambda+1)_n}{\left(\lambda+\frac{1}{2}\right)_n}P_n^{\lambda+1/2,\,\lambda-1/2}(x). \tag{1.18}$$

Also, from

$$C_n^{\lambda+1}(-x)=(-1)^n C_n^{\lambda+1}(x),\quad P_n^{\lambda+1/2,\,\lambda-1/2}(-x)=(-1)^n P_n^{\lambda-1/2,\,\lambda+1/2}(x)$$

and using (1.18), we obtain

$$C_n^{\lambda+1}(x)-C_{n-1}^{\lambda+1}(x)=\frac{(2\lambda+1)_n}{\left(\lambda+\frac{1}{2}\right)_n}P_n^{\lambda-1/2,\,\lambda+1/2}(x), \tag{1.19}$$

so that, from (1.18) and (1.19) it follows

$$C_n^{\lambda+1}(x)=\frac{1}{2}\frac{(2\lambda+1)_n}{\left(\lambda+\frac{1}{2}\right)_n}\left(P_n^{\lambda+1/2,\,\lambda-1/2}(x)+P_n^{\lambda-1/2,\,\lambda+1/2}(x)\right) \tag{1.20}$$

and

$$C_{n-1}^{\lambda+1}(x)=\frac{1}{2}\frac{(2\lambda+1)_n}{\left(\lambda+\frac{1}{2}\right)_n}\left(P_n^{\lambda+1/2,\,\lambda-1/2}(x)-P_n^{\lambda-1/2,\,\lambda+1/2}(x)\right). \tag{1.21}$$

Substituting n for $(n-1)$ in (1.21) and comparing the result with (1.20) we find

$$\begin{aligned}(2\lambda+n+1)&\left(P_{n+1}^{\lambda+1/2,\,\lambda-1/2}(x)-P_{n+1}^{\lambda-1/2,\,\lambda+1/2}(x)\right)\\ &=\left(\lambda+n+\frac{1}{2}\right)\left(P_n^{\lambda+1/2,\,\lambda-1/2}(x)+P_n^{\lambda-1/2,\,\lambda+1/2}(x)\right).\end{aligned} \tag{1.22}$$

It is known that

$$\int_{-1}^{+1}\frac{1-P_n(x)}{1-x}\,dx=2\sum_{r=1}^{n}\frac{1}{r}, \tag{1.23}$$

which can also be written in the form

$$\int_{-1}^{+1}\frac{P_m(x)-P_n(x)}{1-x}\,dx=2\sum_{r=m+1}^{n}\frac{1}{r}\qquad (n\geq m). \tag{1.24}$$

Substituting (1.11) and (1.12) into (1.23) and (1.24) (for $m=1$) respectively we have

$$\int_{-1}^{+1}\sum_{i=1}^{k}\left(2i-\frac{1}{2}\right){}_2F_1\left(-(2i-1),\,2i;\,2;\frac{1-x}{2}\right)dx=\sum_{r=1}^{2k}\frac{1}{r}, \tag{1.25}$$

$$\int_{-1}^{+1}\sum_{i=1}^{k}\left(2i+\frac{1}{2}\right){}_2F_1\left(-2i,\,2i+1;\,2;\,\frac{1-x}{2}\right)dx=\sum_{r=2}^{2k+1}\frac{1}{r}. \tag{1.26}$$

Also, using the relation

$${}_2F_1(a, b; c; z) = (1-z)^{c-a-b}\,{}_2F_1(c-a, c-b; c; z),$$

(1.11) and (1.12) can be transformed into

$$(1.27)\qquad 1-\sum_{i=1}^{k}\Big(2i-\frac{1}{2}\Big)(1-x^2)\,{}_2F_1\Big(-(2i-2), 2i+1; 2; \frac{1-x}{2}\Big)=P_{2k}(x),$$

$$(1.28)\qquad x-\sum_{i=1}^{k}\Big(2i+\frac{1}{2}\Big)(1-x^2)\,{}_2F_1\Big(-(2i-1), 2i+2; 2; \frac{1-x}{2}\Big)=P_{2k+1}(x).$$

From (1.23) and (1.27) we easily deduce

$$(1.29)\qquad \int_{-1}^{+1}\sum_{i=1}^{k}\Big(2i-\frac{1}{2}\Big)(1+x)\,{}_2F_1\Big(-(2i-2), 2i+1; 2; \frac{1-x}{2}\Big)\mathrm{d}x$$

$$=2\sum_{r=1}^{2k}\frac{1}{r}\qquad (k>1).$$

* * *

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